Lec 1: Nilpotent orbits

1) Nilp. orbits & \( \mathfrak{g}_0 \)-tuples
2) Applications of \( \mathfrak{g}_0 \)-tuples
3) Symplectic singulars

4) Nilp. elts & orbits: count simple \( G/U \) (2 @\$ Lie(G)

Thm: Defn: \( x \in g \) is nilp. if follow equiv condns hold

1) \( 0 \in G_0 \) (Zassenhaus topology)
2) \( p(x) \) is nilp. \( \forall x \in g \) \( \Rightarrow \) rep. \( \phi: g \rightarrow g \) (irred)
3) \( \phi \) for some faithful

E.g. if \( g \) is classical, then nilp. elt \( \Rightarrow \) nilp. centralizing

Defn: nilp. orbit: \( O := G_x, x \in g, x \) nilp.

5) \( \mathfrak{g}_0 \)-tuples: Defn: \( \mathfrak{g}_0 \)-tuple in \( g \) is \( (e, h, f) \) w. \( [h, e] = 2e, [h, f] = 2f, h = h \) homot. \( \phi: \mathfrak{g}_0 \rightarrow g \)

Thm (Jacobson-Morozov existence): \( \forall \) nilp. \( x \in g \exists \mathfrak{g}_0 \)-tuple \( (e, h, f) \)

Thm (Kostant, uniqueness): \( (e, h, f), (e', h', f') \) \( \mathfrak{g}_0 \)-tuples \( \Rightarrow \exists \lambda \in g \) \( \forall g \in Z_2(e) \)

Cor: nilp. orbits \( \leftrightarrow \) \# \( \mathfrak{g}_0 \rightarrow \mathfrak{g}/G \) (**)

Morally, it's easier to understand the r.h.s. \( \mathfrak{g}_0 \)-tuple overall is a crucial tool to study various questions about nilpotent orbits

6) Application: classification of nilp. orbits:

\( g = \mathfrak{sl}_n \): (**): \( n \)-dim. \( \mathfrak{g}_0 \)-tuples \( \rightarrow \) Young diagrams (so nilp. orbits are classified by Young diagrams - by Jordan type)

\( g = \mathfrak{so}_n \): \( G = O_n \)-disconnected, compact group is \( SU(n) \) or \( g = \mathfrak{sp}_n \) \( \cdot \) \( n \)-dim. orth/sympl. \( \mathfrak{g}_0 \)-tuples \( \rightarrow \) sympl. 150a (if there's an iso can choose it orthog.

Compare to the fact that up to conj. there's a single orthog/sympl form on \( \mathfrak{g}_0 \):

So our question is: when is \( \mathfrak{g}_0 \)-rep. orthog./sympl.

\( \Rightarrow V \)-vector space \( \Rightarrow \) \( V \otimes V^* \) have n.n.c. orth & sympl. form
\[ E_{\frac{1}{2}} \text{-rep V is orthog if } \dim V \text{ odd, sympl if } \dim V \text{ even } \Rightarrow \text{self-dual} \]

Result: \( V \) is \( E_{\frac{1}{2}} \)-rep; \# even/odd dim E-rep appears in \( V \) \# even mult

\[ \Leftrightarrow \text{V is orthog/sympl} \]

Conclusion: M. If \( O^n \)-orbits in \( S\mathfrak{g}_n / S\mathfrak{p}_n \)-orbits in \( S\mathfrak{g}_n \leftrightarrow \text{Young diagr.} \), where even/odd parts have even mult.

Not'n: \( x^n + n \mapsto O_x \subseteq g \)

Rem: \( O^n \)-orbit \( O_x \) splits into \( 2 \) \( S\mathfrak{g}_2 \)-orbits \( \Leftrightarrow \text{all parts of } x \text{ are even} \).

Otherwise, \( O_x \) is a single \( S\mathfrak{g}_2 \)-orbit.

Gen'l Thm: \# imp. orbits in \( g \leq \infty \)

2.2) Resol'n of sing's for \( \mathcal{O} \)

Not'n: \( g_j := \{ x \in g \mid [h, x] = 0 \} \Rightarrow g_j = \bigoplus g_j \), Lie algebra grading

\( g_{\mathfrak{g}_2} = g_{2,0} \), parabolic subalgebra \( \Rightarrow g_{\mathfrak{g}_2} \leq g_2, \ g_{\mathfrak{g}_2} \cong \mathfrak{g}_2 \).

\( g \times g_{\mathfrak{g}_2} = g \times g_{2,2} \rightarrow (g, x) \sim (g \cdot h, x), \) be \( g_{\mathfrak{g}_2} \), \( [g, x] = \text{equiv. class of } (g, x) \)

\( \tau : [g, x] \mapsto g_x \) (generalized Springer morphism)

Prop. im \( \tau \subset \mathcal{O} \) \& \( \tau : g \times g_{\mathfrak{g}_2} \rightarrow \mathcal{O} \) is resol'n of sing's

Proof: \( \tau \) is project.

\[ g \times g_{\mathfrak{g}_2} \rightarrow g \times g_{\mathfrak{g}_2} \rightarrow g \times g_{\mathfrak{g}_2} \rightarrow g_2 \]

\[ \text{part flag variety} \]

\[ \text{- Lie algebrae are the same } g_{\mathfrak{g}_2} (e) = g \ (e) \text{ - from rep'n theory of } \mathcal{O} \]

\( N \leq Z^e (e) \text{ con comm subgroup w. Lie(N) = g(e)} \text{ & } g_{\mathfrak{g}_2} \text{ - normal subgp} \)

Q: \( Z^e (e) \cap C_0 = Z^e (e, h) = Z^e (e, f) \text{ reductive} \)

\( (x) \leq Z^e (e) = Q \times N \leq Z^e (e) h = Nh \leq e \)

3.1) Simple sing's \( X \) normal alg. vary, \( (X^{alg}) \) sympl. vary

Def'n (Beauville '00) \( X \) has simple sing's if \( \exists (\Rightarrow X) \text{ resol'n of sing's} \)
\( \rho : X \rightarrow X \) s.t. \( \rho^*(w) \) extends from \( \overline{p}''(X^{reg}) \) to \( X \) (may become degenerate)

Remark: Simple sing'ys is Cohen-Macaulay, Gorenstein

32) Example: minim of \( \overline{O} \): \( O \subset \mathfrak{g}^* \). Orbit (for any alg. grp C) is symplectic.

\( T_a \Theta = \mathfrak{g}_a a, \mathfrak{g}_a (x, y, a) = \langle x, y \rangle \) Killing-Kostant form

\( w \) is C-inv & sympl \( \Rightarrow \dim \Theta \) Even

\( \#\) sing'ys \( \Theta \implies \Theta \approx \mathfrak{g}_a \) \( \Rightarrow \Theta \) is symplectic

\( \#\) nilp. orits \( \leq \infty \) \( \Rightarrow \) coding \( \Theta \not\approx \mathfrak{g}_a \), \( \Theta \) may fail to be normal.

\( X = \text{Spec } C[\Theta] \) is norm of \( \Theta \)

Thm (anyushnev) \( X \rightarrow \text{Spec } C[\Theta] \) has sympl. sing's

Proof: \( X = \mathfrak{g}_a \mathfrak{g}_a, \mathfrak{g}_a \rightarrow \Theta \)

\( \overline{\mathfrak{g}} \) is enough to show \( \mathfrak{g} \) exits from \( \mathfrak{g} \subset X \) to \( X \)

\( T_{[x]} \mathfrak{g} \mathfrak{g}_{[y]} = \mathfrak{g}_{[x]} \mathfrak{g}_{[y]} \mathfrak{g}_{[z]} \)

Exerc: \( \exists \mathfrak{g} \)-inv 2-form \( \omega \) on \( X \) \( \omega |_{\mathfrak{g}_{[x]}(y_{\alpha} + y_{\beta}, z_{\alpha} + z_{\beta})} = \) \( (y_{\alpha} + y_{\beta}, z_{\alpha} + z_{\beta}) - (x_{\alpha}, y_{\alpha}). \)\)

\( \omega \) is non-deg \( \Leftrightarrow \mathfrak{g}_{[y]} = 0 \)

Goal of this course: study deformations of \( C[\Theta] \) for nilp. orits \( \Theta \) (if some covers).