Birational geometry of G-varieties

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Abstract

These notes are made to go a little further in the different theories introduced in a four talks lecture during the summer school "Current Topics in the Theory of Algebraic Groups", in Dijon, on July 3rd - 7th, 2017.

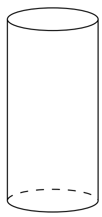
The aim of this lecture was to introduce one of the big theories of complex algebraic geometry, the Minimal Model Program (MMP), in the case of G-varieties, i.e., algebraic varieties endowed with an action of an algebraic group G.

In these notes, I first introduce birational geometry and the MMP, and I explain what happens when an algebraic group acts on the varieties. Secondly, I introduce the theory of Cartier divisors or equivariantly line bundles; and in particular G-linearized line bundles over G-varieties, which naturally reveal a family of G-varieties: spherical varieties. Then, I introduce two particular subfamilies of spherical varieties: toric and horospherical varieties. And I explain how to study the geometry of these varieties by using the theory of algebraic groups and convex geometry (in particular polytopes). Finally, I describe the MMP for horospherical varieties via one-parameter families of polytopes.

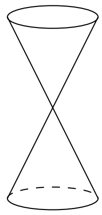
1 A short introduction to birational geometry

Here, **varieties** are algebraic varieties over $\mathbb{C}$, i.e., locally defined as the set of zeros of polynomials. Moreover we suppose them to be irreducible, i.e., they cannot be the union of two varieties. The topology of these varieties is the **Zariski topology**, where closed sets are sets of zeros of polynomials; in particular, non-empty open sets are dense.

Example 1.1. The cylinder: $y^2 + x^2 = 1$.



The cone: $z^2 = y^2 + x^2$.

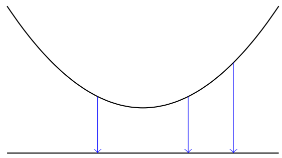


The equation $z = 0$ defines closed subsets: a circle in the cylinder and the union of two lines in the cone (over $\mathbb{C}$).

Morphisms are maps locally defined by polynomials.

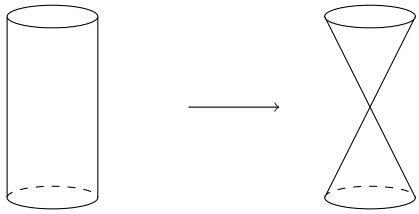
Example 1.2. – The projection of a parabola to a line

$$\begin{array}{ccc} \{(x,y)\mid y=x^2+1\} & \longrightarrow & \{(x,y)\mid y=0\} \\ (x,y) & \longmapsto & (x,0) \end{array}$$



– The natural map from the cylinder to the cone

$$\begin{array}{ccc} \{(x,y,z)\mid x^2+y^2=1\} & \longrightarrow & \{(x,y,z)\mid z^2=x^2+y^2\} \\ (x,y,z) & \longmapsto & (xz,yz,z) \end{array}$$



An **isomorphism** is a bijective morphism whose inverse is also a morphism.
The first map above is an isomorphism.
The second one is an isomorphism on the open sets $z \neq 0$, then we say that the morphism is **birational**.

Definition 1.3. Two varieties X and X' are said to be **birational** if there exists open sets Ω_X and $\Omega_{X'}$ of X and X' respectively such that Ω_X and $\Omega_{X'}$ are isomorphic (i.e., there exists an isomorphism from Ω_X to $\Omega_{X'}$).

Birationality is an equivalence relation. Then it is natural to ask:

- what are the birationality classes?
- is there a “minimal” or “canonical” variety in each class?

From now on, we only consider **projective varieties**, i.e., sets of zeros of homogeneous polynomials in a projective space.

A partial answer to the above questions is given by the **Minimal Model Program** by using birational morphisms. The principle of the MMP is to transform a variety X into a birational variety that is either a **minimal model** (i.e., with only “non-negative” curves) or a **Mori fibration** (a fibration with general fiber a Fano variety i.e., with only “negative” curves).

For this we use morphims that contracts “negative” curves, called **contractions**, and if necessary we use partial desingularizations that add “positive” curves, called **flips**.

There are several types of singularities for which the MMP works, the classical MMP only deals with $\mathbb{Q}$-factorial singularities (Definition 2.15), but we can also consider an MMP with $\mathbb{Q}$-Gorenstein singularities (Definition 2.15): see [Pas14].
We can also consider several types of positivity of curves, but we generally consider the one given by the canonical divisor of X (see Section 2.3).

We summarize the MMP in Figure 1.

Definition 1.4. A **contraction** $\phi : X \longrightarrow Y$ is a projective morphism such that $\phi_*(\mathcal{O}_X) = \mathcal{O}_Y$. A contraction is said to be **small**, if it birational and the complementary of the set of isomorphism of ϕ (i.e., the maximal open set on which ϕ restricts to an isomorphism) is of codimension at least 2.

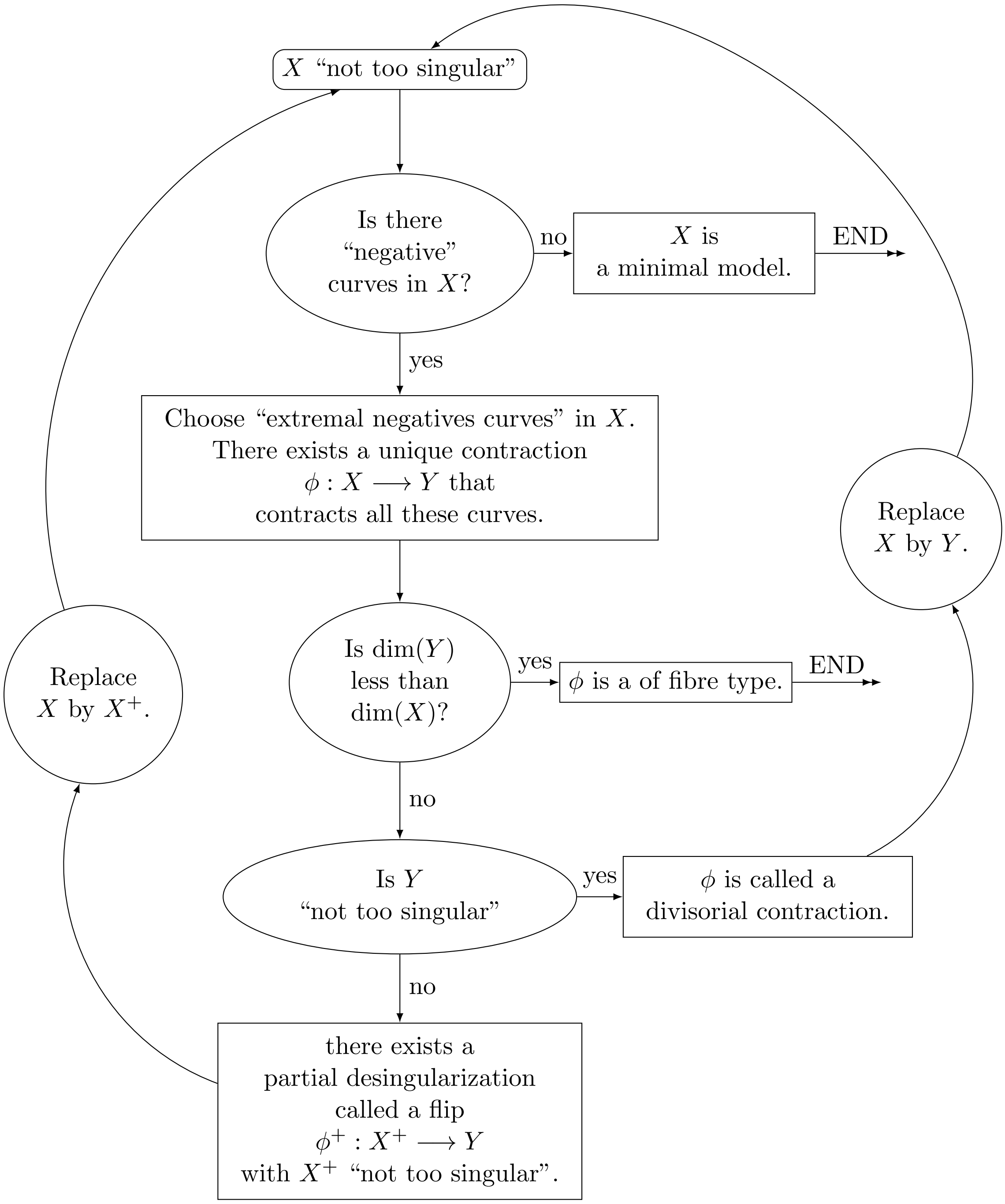
In particular, a contraction is a proper morphism and has connected fibers. Moreover, there is no small contraction in dimension 2.

Definition 1.5. Let $\phi : X \longrightarrow Y$ be a contraction that contracts “negative” curves, with X “not too singular” and Y ”too singular”. A flip associated to ϕ is a small contraction $\phi^+ : X^+ \longrightarrow Y$ that contracts “positive” curves with X^+ “not too singular”.

Proposition 1.6. *If a flip exists, it is unique.*

Theorem 1.7 (Birkar, Cascini, Hacon, McKernan, 2010)**.** *Flips exists.*

Figure 1: the classical MMP



It is still a conjecture that the MMP ends for any choices of "extremal negatives curves".

Consider now the MMP from a G-variety.

We denote by G a connected algebraic group over $\mathbb{C}$, i.e., a group and a variety such that the mutliplication and the inversion are morphisms of varieties. (Note that G is connected if and only if G is irreducible.) We say that G acts on a variety X if there is an action $G \times X \longrightarrow X$ that is a morphism; and then X is called a **G-variety**. If $\phi : X \longrightarrow Y$ is a morphism between two G-varieties, we say that ϕ is G-equivariant if

$$\forall g \in G, \forall x \in X, \phi(g \cdot x) = g \cdot \phi(x).$$

The following result justifies that, without loss of generality, we can consider G-varieties instead of varieties in the MMP.

Lemma 1.8 (Blanchard 1956; Brion, Samuel, Uma, 2013)**.** *Let X and Y be two varieties, let G be a connected algebraic group and let $\phi : X \longrightarrow Y$ be a proper morphism such that $\phi_*(\mathcal{O}_X) = \mathcal{O}_Y$. Suppose that G acts on X. Then there exists a unique action on Y such that ϕ is G-equivariant.*

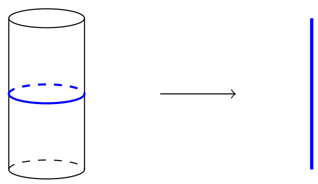
The lemma applies in particular to contractions. If we admit as the same way that, the action of G extends G-equivariantly in flips, we get the following.

Proposition 1.9. *Let X be a ("not too singular") G-variety, then there exist unique G-actions on all varieties constructed in the MMP from X, such that all constructed contractions are G-equivariant.*

Corollary 1.10. *Let X be a ("not too singular") G-variety with an open G-orbit isomorphic to G/H. Then, as long as we do not end with a Mori fibration, the varieties constructed in the MMP from X also have an open G-orbit isomorphic to G/H.*

Proof. As long as we do not have a Mori fibration, all contractions are birational and G-equivariant. In particular, the sets of isomorphism of these contractions are G-stable, and then they must contain the open G-orbit G/H. $\square$

Example 1.11. Let X be the cylinder of Example 1.1. Let $\phi : X \longrightarrow Y$ be a contraction that contracts the horizontal circle at $z = 0$ (i.e., sends it to a point). Let the additive group $\mathbb{G}_a \simeq \mathbb{C}$ acts on X by vertical translation $(\lambda \cdot (x, y, z) \longmapsto (x, y, z + \lambda))$. Then the lemma implies that all horizontal circles are also sent to a point by ϕ. Then Y is the vertical line and ϕ is the natural projection.



We can also consider the projectivization of X defined by $\bar{X} := \{[x, y, z, w] \in \mathbb{P}^3 \mid x^2+y^2+w^2 = 0\}$. Then $\mathbb{G}_a$ also acts on $\bar{X}$ by vertical translation ($\lambda\cdot[x, y, z, w] \longmapsto [x, y, z+\lambda w, w]$). As before, we deduce that $\dim(Y) \leq 1$ and then that the two lines at infinity are also contracted by ϕ.

But $\mathrm{SO}_3(\mathbb{C})$ (or $\mathrm{SL}_2(\mathbb{C})$) also acts on $\bar{X}$ (action deduced from the action on $\mathbb{C}^3 \simeq \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}w$). Then a vertical line ($x = 0$ and $w = iy$ for example) is sent by an element of $\mathrm{SO}_3(\mathbb{C})$ to one of the two lines at infinity $w = 0$ and $x = iy$. Hence, any vertical line is also contracted by ϕ, and then Y is a point.

We will see later in Example 4.4 that there exists another projectivization of X that admits a contraction to $\mathbb{P}^1$.

2 A short introduction to divisors and line bundles

We first recall briefly the theory of divisors and line bundles. All varieties are supposed to be projective and normal (in particular regular in codimension one). We also introduce the anticanonical line bundle, which plays an important role in the MMP. And finally, we specialize the theory to G-linearized line bundles.

2.1 The general theory of divisors and line bundles

Projective varieties can be embedded by different ways in different projective spaces. For exemple, $\mathbb{P}^1$ can be embedded in itself or in $\mathbb{P}^2$ by the embedding $[x, y] \longmapsto [x^2, xy, y^2]$. These different ways are parametrized by geometric objects called very ample line bundles. Line bundles, invertible sheaves and Cartier divisors are equivalent in our context where varieties are normal. See [Har77] or [Per08] for more details on this theory.

We denote by $\mathbb{C}(X)$ the field of rational functions on X, i.e., the fraction field of the ring of functions of any affine open set of X.

Example 2.1. Let $X = \mathbb{P}^n$. Then $\mathbb{C}(X)$ is $\mathbb{C}(X_0, X_1, \dots, X_n)_{\mathrm{hom}}$ i.e., the fraction field of the polynomial ring $\mathbb{C}[X_1/X_0, \dots, X_n/X_0]$, which is the ring of functions of the affine open set $U_0 := \{[x_0, \dots, x_n] \in \mathbb{P}^n \mid x_0 \neq 0\}$.

Definition 2.2. An **prime divisor** of X is an irreducible subvariety of X of codimension 1. A Weil divisor of X is a formal sum of prime divisors with coefficients in $\mathbb{Z}$.

To a function f in $\mathbb{C}(X)$, we associate a Weil divisor $\operatorname{div}(f)$ as follows: for any prime divisor D of X, the coefficient of D in $\operatorname{div}(f)$ is the degree of zeros of f along D (or poles and then the coefficient is negative). We say that two Weil divisors D and D' are **linearly equivalent** if there exists f in $\mathbb{C}(X)$ such that $D = D' + \operatorname{div}(f)$.

Example 2.3. Let $X = \mathbb{P}^n$ and $f = x_1 x_2^2 / x_0^3$. Then $\operatorname{div}(f) = D_1 + 2D_2 - 3D_0$, where $D_i := \{[x_0, \dots, x_n] \in \mathbb{P}^n \mid x_i = 0\}$.

Definition 2.4. A Weil divisor Δ of X is said to be a **Cartier divisor** if it is locally linearly equivalent to zero: for any $x \in X$, there exist an open set U of X containing x and a rational function f_U in $\mathbb{C}(U)$ such that $\Delta_{|U} = \operatorname{div}(f_U)$.

Note that the function f_U is only unique up to multiplication by an invertible function on U. Moreover there exists a finite covering of X by such open sets.

For normal varieties, Cartier divisors are “in bijection” with line bundles.

Definition 2.5. Let Δ be a Cartier divisor of X, with functions f_U associated to a finite covering of X as above. Then we define the line bundle $\mathcal{L}(\Delta)$ to be the locally trivial line bundle on X defined with the following transition maps for any U and V in the open covering

$$\begin{array}{cccc} \phi_{UV}: & U \cap V \times \mathbb{C} & \longrightarrow & U \cap V \times \mathbb{C} \\ & (u, z) & \longmapsto & (u, \frac{f_V}{f_U}(u) z). \end{array}$$

To get the reverse construction from a line bundle $\mathcal{L}$ on X, choose a finite covering of X that locally trivializes $\mathcal{L}$. Then fix one of the open set of the covering, and then there exists a unique divisor Δ such that for any U in the covering $\Delta_{|U} = \operatorname{div}(\phi_{U_0 U})$, where $\phi_{U_0 U}$ are the transition maps.

Example 2.6. Let $X = \mathbb{P}^n$ and $\Delta = D_0$, with notation of Examples 2.1 and 2.3. We consider the canonical covering $\bigcup_{i=0}^n U_i$ of $\mathbb{P}^n$. Then for any $i \in \{0, \dots, n\}$, $\Delta_{|U_i} = \operatorname{div}(x_0/x_i)$. And $\mathcal{L}(D_0)$ is defined by transition maps

$$\begin{array}{cccc} \phi_{ij}: & U_i \cap U_j \times \mathbb{C} & \longrightarrow & U_i \cap U_j \times \mathbb{C} \\ & ([x_0, \dots, x_n], z) & \longmapsto & ([x_0, \dots, x_n], \frac{x_j}{x_i} z). \end{array}$$

The line bundle $\mathcal{L}(D_0)$ on $\mathbb{P}^n$ is denoted by $\mathcal{O}(1)$ or $\mathcal{O}_{\mathbb{P}^n}(1)$.

The tautological line bundle $\{([x], z) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid z \in \mathbb{C}x\}$ is the line bundle $\mathcal{L}(-D_0)$.

Definition 2.7. Let Δ be a Cartier divisor of X, with notation of Definition 2.4. A global section of Δ is a family $(s_U)_U$ of regular functions s_U on U, such that for any U and V, $s_V = \frac{f_V}{f_U} s_U$ on $U \cap V$. It defines a section $s : X \longrightarrow \mathcal{L}$ of $\mathcal{L}(\Delta)$ (by for any $x \in U$, $s(x) = (x, s_U(x)) \in U \times \mathbb{C}$), and inversely.

We denote by $H^0(X, \Delta)$ or $H^0(X, \mathcal{L})$ the set of global sections of divisors and line bundles. It is a vector space over $\mathbb{C}$.

Example 2.8. Let $X = \mathbb{P}^n$, with notation of Examples 2.1, 2.3 and 2.6. A basis of $H^0(X, D_0)$ is $(s_0, \dots, s_n)$ where s_i is the family $(x_i/x_j)_{j \in \{0, \dots, n\}}$ (associated to the canonical covering of $\mathbb{P}^1$).

And $H^0(X, -D_0) = 0$.

Definition 2.9. A Cartier divisor Δ of X is said **globally generated** (or base point free) if for any $x \in X$ there exists a section s of Δ such that s does not vanish in x (i.e., the exists U such that $x \in U$ and $s_U(x) \neq 0$, or equivalently for any U, $s_U(x) \neq 0$).

Proposition 2.10. *Let Δ be a globally generated Cartier divisor of X. Then the map*

$$\begin{array}{rccc} \phi_\Delta: & X & \longrightarrow & \mathbb{P}(H^0(X,\Delta)^*) \\ & x & \longmapsto & [s \longmapsto s(x)] \end{array}$$

is well-defined.

Proof. We just have to prove that the element $[s \longmapsto s_U(x)]$ in $\mathbb{P}(H^0(X,\Delta)^*)$ does not depend on the open set U containing x. But if U and V contain x, then $s_V(x) = \frac{f_V}{f_U}(x)s_U(x)$ where $\frac{f_V}{f_U}(x)$ is a non-zero complex number that does not depend on s. $\square$

Example 2.11. If $\Delta = 0$ then $H^0(X,\Delta) = \mathbb{C}$ and $\phi_\Delta : X \longrightarrow \mathbb{P}^0 \simeq \mathrm{pt}$.

Example 2.12. Let $X = \mathbb{P}^n$, with notation of Examples 2.1, 2.3, 2.6 and 2.8. The divisor D_0 is globally generated because for any $[x_0,\dots,x_n] \in \mathbb{P}^n$, there exists $i \in \{0,\dots,n\}$ such that $x_i \neq 0$ and then s_i does not vanish in x. And the map ϕ_{D_0} is an isomorphism.

The divisor $2D_0$ is also globally generated, $H^0(X,2D_0) = S^2\mathbb{C}^{n+1}$ and ϕ_{2D_0} is the Veronese embedding:

$$\begin{array}{ccc} \mathbb{P}^n & \longrightarrow & \mathbb{P}(S^2\mathbb{C}^{n+1}) \\ [x_0,\dots,x_n] & \longmapsto & [x_0^2, x_0x_1,\dots,x_ix_j,\dots,x_n^2] \end{array}$$

In that latter example, the maps ϕ_Δ are embeddings of X in projective spaces, which are the cases we are interested in.

Definition 2.13. Let Δ be a Cartier divisor of X. We say that Δ is **very ample** if ϕ_Δ is an embedding. We say that Δ is **ample** if there exists a positive integer k such that $k\Delta$ is very ample.

We can give the same definition with line bundles instead of Cartier divisors.

Proposition 2.14. *Let ϕ be an embedding of X in $\mathbb{P}^n$, such that $\phi(X)$ is contained in no linear subspace of $\mathbb{P}^n$. Then the line bundle $\phi^*(\mathcal{O}(1))$ (i.e., the restriction of $\mathcal{O}(1)$ to X) is very ample. And the morphism ϕ is the one associated to $\phi^*(\mathcal{O}(1))$.*

Proof. A global section of $\phi^*(\mathcal{O}(1))$ is the restriction of a global section of $\mathcal{O}(1)$, then The vector space of global sections of $\phi^*(\mathcal{O}(1))$ is generated by the restrictions $s_{i|X}$ of the section s_i of $\mathcal{O}(1)$ defined in Example 2.8. Hence $\phi^*(\mathcal{O}(1))$ is clearly globally generated because $\mathcal{O}(1)$ is, so that the morphism ϕ' associated to $\phi^*(\mathcal{O}(1))$ is well-defined. The $s_{i|X}$ are linearly independent because $\phi(X)$ is contained in no linear subspace of $\mathbb{P}^n$. Then ϕ' also goes to $\mathbb{P}^n$ and it is not difficult to check that $\phi' = \phi$. $\square$

Definition 2.15. A $\mathbb{Q}$-divisor D is a formal sum of prime divisors with coefficients in $\mathbb{Q}$. A divisor (or a $\mathbb{Q}$-divisor) is said to be $\mathbb{Q}$-Cartier if some multiple is Cartier.

A variety is said to be $\mathbb{Q}$-factorial if any Weil divisor is $\mathbb{Q}$-Cartier.
A variety is said to be $\mathbb{Q}$-Goresntein if its canonical divisor (defined below) is $\mathbb{Q}$-Cartier.

2.2 Anticanonical divisors

An important and natural line bundle of a smooth variety X of dimension d is the determinant $\bigwedge^d T_X$ of its tangent bundle, called the **anticanonical line bundle**. An **anticanonical divisor** of X is a divisor associated to the anticanonical line bundle (unique up to linear equivalence). We denote it by $-K_X$. (The divisor K_X is called **canonical divisor**.)

If X is not smooth but normal, we can extend the definition of anticanonical divisor by taking the anticanonical divisor of the smooth part of X and extending it to X (in a unique way because X is regular in codimension one).

In the rest of the paper we use the theory of intersection of curves with lines bundles (or equivalently Cartier divisors). The intersection of a divisor with a curve would be intuitively defined as the number of points (with multiplicity) of their intersection (the intersection of a one-codimensional subvariety with a one-dimensional subvariety is zero-dimensional in general). Let me give here a more precise definition in the particular case wher the curve is isomorphic to $\mathbb{P}^1$.

Definition 2.16. Let X be a variety and $\mathcal{L}$ be a line bundle over X. Let C be a curve in X that is isomophic to $\mathbb{P}^1$. Then the restriction $\mathcal{L}_{|C}$ of $\mathcal{L}$ to C is a line bundle over $\mathbb{P}^1$ and then $\mathcal{L}_{|C} \simeq \mathcal{O}_{\mathbb{P}^1}(k)$ for a certain $k \in \mathbb{Z}$. The integer k si called the **intersection number** of $\mathcal{L}$ with C and denoted by $\mathcal{L} \cdot C$.

In that paper, if K_X is Cartier, we say that a curve C is **positive** (resp. **negative**, resp. zero) if $K_X \cdot C$ is positive (resp. negative, resp. zero).

2.3 G-linearized line bundles

In this section, G is a linear algebraic group over $\mathbb{C}$.

Definition 2.17. Let X be a G-variety and $\mathcal{L}$ be a line bundle over X. Denote $\pi : \mathcal{L} \longrightarrow X$. Then $\mathcal{L}$ is said to be **G-linearized** if G acts on $\mathcal{L}$ so that π is G-equivariant and G acts linearly on each fiber, i.e., for any $g \in G$ and any $l \in \mathcal{L}$, we have $\cdot\pi(g \cdot l) = g \cdot \pi(l)$, and the map from the line $\pi^{-1}(\pi(l)) \simeq \mathbb{C}$ to the line $\pi^{-1}(\pi(g \cdot l)) \simeq \mathbb{C}$ that sends l' to $g \cdot l'$ is linear.

Proposition 2.18. *[KKLV89, Section 2.4] Let X be a normal G-variety and $\mathcal{L}$ be a line bundle over X.*

Then, there exists a positive integer n such that $\mathcal{L}^{\otimes n}$ is G-linearized.

Moreover n can be choosen to be the order of the Picard group of G. In particular, if $\tilde{G}$ denotes the universal cover of G, then $\mathcal{L}$ is $\tilde{G}$-linearized.

Example 2.19. Let $G = \mathrm{PSL}_{n+1}(\mathbb{C})$ acting on $\mathbb{P}^n$. Consider the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{P}^n$. Recall that it is defined by $\{([x], z) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid z \in \mathbb{C}x\}$. Then $\tilde{G} = \mathrm{SL}_{n+1}(\mathbb{C})$ acts naturally on $\mathcal{O}(-1)$ so that $\mathcal{O}(-1)$ is $\tilde{G}$-linearized. Also the tautological line bundle is not G-linearized, but $\mathcal{O}(-n-1)$ is G-linearized. Indeed, $\mathcal{O}(-n-1)$ is defined by $\{([x], z) \in \mathbb{P}^n \times S^{n+1}\mathbb{C}^{n+1} \mid z \in \mathbb{C}\, x \otimes \cdots \otimes x\}$ and then has a natural action of G.

Proposition 2.20. *Let X be a normal G-variety and $\mathcal{L}$ be a G-linearized line bundle over X. Then the vector space $H^0(X, \mathcal{L})$ is a G-module.*

Proof. The vector space $H^0(X, \mathcal{L})$ is the space of global sections $s : X \longrightarrow \mathcal{L}$. Then, G acts on $H^0(X, \mathcal{L})$ by: for any $g \in G$, for any $s \in H^0(X, \mathcal{L})$ and for any $x \in X$ we have $g \cdot s(x) = g \cdot s(g^{-1} \cdot x)$. (The map $g \cdot s$ satisfies $\pi \circ (g \cdot s) = \mathrm{id}$ because π is G-equivariant.) Moreover G acts linearly on $H^0(X, \mathcal{L})$ because it acts linearly on each fiber of $\mathcal{L}$. $\square$

Proposition 2.21. *Let X be a normal G-variety and $\mathcal{L}$ be a G-linearized globally generated line bundle over X. Then the map*

$$\begin{array}{cccc} \phi_{\mathcal{L}} & X & \longrightarrow & \mathbb{P}(H^0(X, \mathcal{L})^*) \\ & x & \longmapsto & [s \longmapsto s(x)] \end{array}$$

is G-equivariant.

Proof. Note that for any $g \in G$, any $s \in H^0(X, \mathcal{L})$ and any $x \in X$, we have $g \cdot [s \longmapsto s(x)] = [s \longmapsto (g^{-1} \cdot s)(x)] = [s \longmapsto g^{-1} \cdot s(g \cdot x)] = [s \longmapsto s(g \cdot x)]$. $\square$

Example 2.22. Let P be a parabolic subgroup of G and χ be a character of P. Then we define the line bundle $\mathcal{L}_\chi$ over G/P by the quotient $G \times^P \mathbb{C}_\chi$ of $G \times \mathbb{C}$ by the equivalence $\sim$ defined as follows:

$$\forall g \in G, \forall p \in P, \forall z \in \mathbb{C}, \, (g, z) \sim (gp, \chi(p)z).$$

(The map π from $\mathcal{L}_\chi$ to G/P is defined by $\pi(\overline{(g, z)}) = gP$.) The group G acts on $\mathcal{L}_\chi$ by left multiplication and π is G-equivariant. Moreover, for any $g \in G$ and any $\overline{(h, z)} \in \mathcal{L}_\chi$ we have $\pi^{-1}(\pi(\overline{(h, z)}) = hP \times^P \mathbb{C}_\chi \simeq \mathbb{C}$ and G acts trivially on each fiber. Any global section of $\mathcal{L}_\chi$ is given by a regular function f of G that satifies: for any $p \in P$ and any $g \in G$, $f(gp) = \chi(p)f(g)$. Indeed, such a function f defines a section $s : G/P \longrightarrow \mathcal{L}_\chi$ by $s(gP) := \overline{(g, f(g))}$. Thus $H^0(X, \mathcal{L}_\chi)^*$ is the irreducible G-module associated to χ (in particular $\{0\}$ if χ is not dominant).

In particular, if $G = \mathrm{SL}_4(\mathbb{C})$, G/P_2 is the Grassmannian of planes in $\mathbb{C}^4$, and $\chi = \varpi_2$ is the fundamental weight of P_2, then the map $\phi_{\mathcal{L}_\chi}$ is the Plücker embedding of G/P_2 in $\mathbb{P}(\bigwedge^2 \mathbb{C}^4)$.

Remark 2.23. The G-module $H^0(X, \mathcal{L})$ is said to be multiplicity free if for any irreducible G-module V, $\mathrm{Hom}_G(H^0(X, \mathcal{L}), V)$ is trivial or one-dimensional, i.e., the multiplicity of V in $H^0(X, \mathcal{L})$ is zero or one.

If for any line bundle $\mathcal{L}$ over the G-variety X, the G-module $H^0(X,\mathcal{L})$ is multiplicity free then X is **spherical**, i.e. has an open orbit under the action of a Borel subgroup B of G.
A particular case of spherical varieties is the case of **horospherical varieties** when the unipotent radical U of B fixes a point x_0 of the G-open orbit of X. In that case, for any very ample line bundle $\mathcal{L}$ over X the map $\phi_{\mathcal{L}}$ maps x_0 to the line generated by a sum of highest weight vectors in $\mathbb{P}(H^0(X,\mathcal{L})^*)$. And, X is isomorphic to the closure of $G\cdot\phi_{\mathcal{L}}(x_0)$ in $\mathbb{P}(H^0(X,\mathcal{L})^*)$. (See for example the case of the Grassmannian above.)

3 Toric and horospherical varieties

The varieties are still supposed to be normal, but not necessarily projective here.

A famous subfamily of horospherical varieties (in addition to flag varieties) is the family of **toric varieties**, when $G=(\mathbb{C}^*)^n$ and $G\cdot x_0\simeq G$ (with notation of Remark 2.23).

We first introduce the theory of toric varieties (see [Ful93] or [CLS11] for more detail). And then we will explain how to generalize the theory to horospherical varieties (see [Pas06], [Pas08] or [Pas15] for more detail).

3.1 Toric varieties

Here $G=(\mathbb{C}^*)^n$ (we use G for the group and $(\mathbb{C}^*)^n$ for the homogeneous space $G\cdot x_0$). We denote by M the set of characters of G (isomorphic to $\mathbb{Z}^n$). We denote $N:=\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$, $M_{\mathbb{Q}}:=M\otimes_{\mathbb{Z}}\mathbb{Q}$ and $N_{\mathbb{Q}}:=N\otimes_{\mathbb{Z}}\mathbb{Q}$. Then the ring of regular functions (and the field of rational functions) on $(\mathbb{C}^*)^n$ is generated by the monomials $W^m:=X_1^{m_1}\cdots X_n^{m_n}$ with $m=(m_1,\dots,m_n)\in M\simeq\mathbb{Z}^n$. For any $m\in M$, the monomial W^m is a function of weight m, i.e. $\forall z\in G$, $\forall x\in(\mathbb{C}^*)^n$, $W^m(z\cdot x)=m(z)W^m(x)$. It is unique up to scalar multiplication. The set of such functions is denoted by $\mathbb{C}((\mathbb{C}^*)^n)^{(G)}$, here $\mathbb{C}((\mathbb{C}^*)^n)^{(G)}=\bigoplus_{m\in M}\mathbb{C}W^m$ and note that $M\simeq\mathbb{C}((\mathbb{C}^*)^n)^{(G)}/\mathbb{C}^*$.

Let X be an affine toric G-variety (of dimension n). Then G acts on the ring $\mathbb{C}[X]$, and we have $\mathbb{C}[X]=\bigoplus_{m\in M}\mathbb{C}[X]_m^{(G)}\subset\bigoplus_{m\in M}\mathbb{C}W^m$. Hence, $\mathbb{C}[X]$ is completely determined by the semi-group $\{m\in M\mid\mathbb{C}[X]_m^{(G)}\neq 0\}$. The fact that X is normal implies that this semi-group is saturated, i.e., it is the intersection of a cone $\sigma^\vee$ in $M_{\mathbb{Q}}$ with M. Denote by σ the dual cone of $\sigma^\vee$ in $N_{\mathbb{Q}}$ ($\sigma=\{n\in N_{\mathbb{Q}}\mid\forall m\in\sigma^\vee,\ \langle m,n\rangle\geq 0\}$). The cone σ is called the cone associated to X, denoted by σ_X.

Now, let X be a toric G-variety. Then it is covered by open G-stable affine subsets. The fan of X, denoted by $\mathbb{F}_X$, is the set of cones σ_U associated to open G-stable affine subsets U of X. It satifies (the definition of fan):

- any cone in $\mathbb{F}_X$ contains no line;
- any face of any cone in $\mathbb{F}_X$ is in $\mathbb{F}_X$;
- any two cones intersect along a commun face.

Example 3.1. Let $X=\mathbb{P}^n$ where G acts by $(z_1,\dots,z_n)\cdot[x_0,\dots,x_n]=[x_0,z_1x_1,\dots,z_nx_n]$. Its open G-orbit is the G-orbit of $[1,\dots,1]$. The monomials W^m are the rational functions of $\mathbb{P}^n$ that sends $[x_0,\dots,x_n]$ to $\frac{x_1^{m_1}\cdots x_n^{m_n}}{x_0^{m_1+\cdots+m_n}}$. The open G-stable affine subvarieties of $\mathbb{P}^n$ are the sets $U_I:=\{[x_0,\dots,x_n]\mid\forall i\in I,\ x_i\neq 0\}$, with non-empty $I\subset\{0,\dots,n\}$. But $\mathbb{C}[U_I]$ is generated by W^m such that $\forall i\notin I$, $m_i\geq 0$ and if $0\notin I$, $m_1+\cdots+m_n\leq 0$; then σ_{U_I} is generated by the e_i with $i\notin I$ and $-e_1-\cdots-e_n$ if $0\notin I$ (where $(e_1,\dots,e_n)$ denotes the canonical basis of N). For example, $\sigma_{U_{\{0,\dots,n\}}}=\{0\}$ (note that $U_{\{0,\dots,n\}}$ is the open G-orbit), $\sigma_{U_{\{1,\dots,n\}}}=\mathbb{Q}_{\geq 0}(-e_1-\cdots-e_n)$, $\sigma_{U_{\{0,2,\dots,n\}}}=\mathbb{Q}_{\geq 0}e_1$, σ_{U_0} is generated by $e_1,\dots,e_n$, and σ_{U_1} is generated by $e_2,\dots,e_n$ and $-e_1-\cdots-e_n$.

Proposition 3.2. *Let X be a toric variety. Every Weil divisor of X is linearly equivalent to a G-stable Weil divisor.*

Prime G-stable divisors of X are in bijection with the set of edges of the fan $\mathbb{F}_X$. We denote them by $X_1,\dots,X_k$, and we denote by $x_1,\dots,x_n$ the primitive element of the associated edges (they are elements of N).

Cartier G-stable divisors of X are in bijection with piecewise linear functions on $\mathbb{F}_X$ whose linear functions are given by elements of M. To a piecewise linear function h on $\mathbb{F}_X$ corresponds the divisor $\sum_{i=1}^k h(x_i)X_i$.

Sketch of proof. The open set $X\backslash\bigcup_{i=1}^k X_i$ is the open G-orbit of X. But $(\mathbb{C}^*)^n$ is factorial, then every Weil divisor of $(\mathbb{C}^*)^n$ is linearly equivalent to 0, and then every Weil divisor of X is linearly equivalent to a linear combinaison of the X_i's.

Let D be a prime G-stable divisor of X. The valuation associated to D restricts to $\mathbb{C}((\mathbb{C}^*)^n)^{(G)}$ and then defines a linear map from M to $\mathbb{Z}$, i.e., an element of N. This element is primitive becauce D is prime, and we can prove that this element generates the cone associated to the open set consisting of the open G-orbit of X and the open G-orbit of D.
Reciprocally, let σ be an edge of $\mathbb{F}_X$ generated by the primitive element n. Then it corresponds to an open G-stable subset U of X, such that $\mathbb{C}[U]$ is generated by W^m with $m\in M$ such that $\langle m,n\rangle\geq 0$. In particular U is isomorphic to $(\mathbb{C}^*)^{n-1}\times\mathbb{C}$, has two orbits: the open G-orbit of X and another one of codimension one whose closure in X is then a prime divisor.

Let $D=\sum_{i=1}^k d_iX_i$ be a Cartier G-stable divisor of X. It is locally principal, more precisely it has to be principal on open G-stable affine subsets of X. Then for any such subset U, corresponding to a cone σ in $\mathbb{F}_X$, we have $D_{|U}=\mathrm{div}(f_U)$ where $f_U\in\mathbb{C}(U)^{(G)}$ (because D is G-stable). But f_U corresponds to an element m_σ in M, which can be viewed as a linear function on $N_\mathbb{Q}$. Then we can check that the family $(m_\sigma)_{\sigma\in\mathbb{F}_X}$ defines a piecewise linear functions on $\mathbb{F}_X$. □

Proposition 3.3. *Let X be a toric variety and $D=\sum_{i=1}^k d_iX_i$ be a Cartier G-stable divisor of X. Then $H^0(X,D)=\bigoplus_{m\in Q\cap M}\mathbb{C}W^m$, where*

$$Q:=\{m\in M_\mathbb{Q}\mid\forall i\in\{1,\dots,k\},\,\langle m,x_i\rangle\geq -d_i\}.$$

Sketch of proof. Let $m\in Q\cap M$. Define s_m by $(s_m)_U:=W^mf_U$ with the notation of the previous proof. We can check that $(s_m)_U$ is a regular function on U because, for any $i\in\{1,\dots,k\}$, $\langle m,x_i\rangle\geq -d_i$.
Reciprocally, since $H^0(X,D)$ is a G-module it decomposes into the direct sum of G-stable lines. Let s be a non-zero element of such a line and denote by m its weight. Then $\frac{s_U}{f_U}$ is W^m up to scalar multiplication, and since s_U is regular on U (for any U), we have for any $i\in\{1,\dots,k\}$, $\langle m,x_i\rangle\geq -d_i$. □

We now give two classical results without proof.

Proposition 3.4. *Let X be a toric variety, D be a Cartier G-stable divisor of X and h_D be the associated piecewise linear function.*
Then D is globally generated if and only if h_D is convex.
And D is ample if and only if h_D is strictly convex.

Example 3.5. Let $X=\mathbb{P}^n$ as in Example 3.1. Then X has $n+1$ prime G-stable divisors: $X_i:=\{[y_0,\dots,y_n]\mid y_i=0\}$, for $i\in\{0,\dots,n\}$. Any Weil divisor is Cartier. We have $H^0(X,lX_0)=\bigoplus_{m\in Q\cap M}\mathbb{C}W^m$ where Q is the simplex of vertices $0,le_1,\dots,le_n$.

Proposition 3.6. *An anticanonical divisor of a toric variety X is the sum $\sum_{i=1}^k X_i$ of all G-stable prime divisors of X.*

Example 3.7. Let $X=\mathbb{P}^n$ as in Example 3.5. An anticanonical divisor of X is $\sum_{i=0}^n X_i$, linearly equivalent to $(n+1)X_0$. In fact, it is well known that the anticanonical line bundle of $\mathbb{P}^n$ is $\mathcal{O}(n+1)$.

3.2 Horospherical varieties

The theory of horospherical varieties is similar to the theory of toric varieties, they are classified by colored fans instead of fans.

Here G is any reductive connected algebraic group over $\mathbb{C}$.

We fix a horospherical G/H-embedding, i.e., such that H contains a maximal unioptent subgroup U of G.

Fix a Borel subgroup B of G such that U is the unipotent radical of B, and T a maximal torus in B. Then denote by P the normalizer of H in G, it is a parabolic subgroup of G containing B and we have a torus fibration $G/H\longrightarrow G/P$. Denote by M the set of characters of P whose resctriction to H is trivial. In particular, $M=\mathbb{C}(G/H)^{(B)}/\mathbb{C}^*$. Keep the same notation for N, $M_\mathbb{Q}$ and $N_\mathbb{Q}$ as in Section 3.1.

Let $\mathcal{R}$ be the set of simple roots of (G,B,T) that are not simple roots of P. For any $\alpha\in\mathcal{R}$, we denote by D_α the B-stable divisor of G/H defined by inverse image of the B-stable Schubert divisor of G/P associated to α. The B-stable divisor D_α, for $\alpha\in\mathcal{R}$, are called **colors**. Any D_α defined a B-invariant valuation on $\mathbb{C}(G/H)$, thus defines an element of N.

In fact the element associated to D_α is the restriction to M of the cocharacter $\alpha^\vee$, it is denoted by $\alpha^\vee_M$ and called the **image of the color** D_α.

Definition 3.8. 1. A **colored cone** in $N_\mathbb{Q}$ is a couple $(\mathcal{C},\mathcal{F})$ where $\mathcal{F}$ is a subset of $\mathcal{S}_P$ and $\mathcal{C}$ is a cone in $N_\mathbb{Q}$ generated by finitely many elements of N and the $\alpha^\vee_M$ with α in $\mathcal{F}$ such that, $\alpha^\vee_M\neq 0$ for any $\alpha\in\mathcal{F}$ and $\mathcal{C}$ contains no line.

2. A colored face of a colored cone $(\mathcal{C},\mathcal{F})$ is a couple $(\mathcal{C}',\mathcal{F}')$ where $\mathcal{C}'$ is a face of the cone $\mathcal{C}$ and $\mathcal{F}'=\{\alpha\in\mathcal{F}\mid\alpha^\vee_M\in\mathcal{C}'\}$. It is in particular a colored cone.

3. A **colored fan** in $N_\mathbb{Q}$ is a finite set $\mathbb{F}$ of colored cones such that: any colored faces of a colored cone of $\mathbb{F}$ is in $\mathbb{F}$, and for any $u\in N_\mathbb{Q}$ there exists at most one colored cone $(\mathcal{C},\mathcal{F})$ of $\mathbb{F}$ such that u is in the relative interior of $\mathcal{C}$.

4. A fan is complete if $\bigcup_{(\mathcal{C},\mathcal{F})\in\mathbb{F}}\mathcal{C}=N_\mathbb{Q}$.

Here, as we do not presice what isomorphism of G/H-embedding are, we consider that a G/H-embedding is just a normal G-variety with an open G-orbit isomorphic to G/H.

Theorem 3.9 ([Kno91]). *There exists a bijective map from the set of isomorphic classes of G/H-embeddings to the set of colored fans in $N_\mathbb{Q}$.*

Moreover, X is complete if and only if its colored fan $\mathbb{F}_X$ is complete.

We denote by $\mathcal{F}_X$ the set of colors of X, i.e. $\bigcup_{(\mathcal{C},\mathcal{F})\in\mathbb{F}}\mathcal{F}$.

Definition 3.10. The **pseudo-moment polytope** of (X,D) is

$$\tilde{Q}_D:=\{m\in M_\mathbb{Q}\mid\langle m,x_i\rangle\geq -d_i,\ \forall i\in\{1,\dots,r\}\text{ and }\langle m,\alpha^\vee_M\rangle\geq -d_\alpha,\ \forall\alpha\in\mathcal{R}\}.$$

Let $v^0:=\sum_{\alpha\in\mathcal{R}}d_\alpha\varpi_\alpha$. The **moment polytope** of (X,D) is $Q_D:=v^0+\tilde{Q}_D$.

Theorem 3.11. *[Bri89] Let X be a G/H-embedding. Denote by $X_1,\dots,X_k$ the G-stable prime divisors or X, and denote by $x_1,\dots,x_k$ the primitive elements of the associated edges (without color) of $\mathbb{F}_X$.*

1. *Any Weil divisor D of X is linearly equivalent to a B-stable divisor, i.e. of the form $D=\sum_{i=1}^k d_iX_i+\sum_{\alpha\in\mathcal{R}}d_\alpha D_\alpha$.*

2. *Such a divisor D is Cartier if and only if for any $(\mathcal{C},\mathcal{F})\in\mathbb{F}_X$ there exists $m\in M$ such that for any $x_i\in\mathcal{C}$, $\langle\chi,x_i\rangle=d_i$ and for any $\alpha\in\mathcal{F}$, $\langle\chi,\alpha^\vee_M\rangle=d_\alpha$. Thus, it defines a piecewise linear function h_D.*

   The G-module $H^0(X,D)$ equals $\bigoplus_{m\in\tilde{Q}_D\cap M}V(v^0+m)$, where $V(\chi)$ denotes the irreducible G-module of highest weight χ.

3. *The Cartier divisor D is globally generated (respectively ample) if and only if h_D is convex (respectively strictly convex) and for any $\alpha\in\mathcal{R}\backslash\mathcal{F}_X$, $h_D(\alpha^\vee_M)\leq d_\alpha$ (respectively $h_D(\alpha^\vee_M)<d_\alpha$).*

4. *An anticanonical divisor of X is*

$$-K_X=\sum_{i=1}^k X_i+\sum_{\alpha\in\mathcal{R}}a_\alpha D_\alpha,$$

   where $a_\alpha=\langle 2\rho^P,\alpha^\vee\rangle$ such that ρ^P is the sum of positive roots of (G,B,T) that are not roots of P.

Note that the a_α are integers greater than or equal to 2.

Morphisms between horospherical varieties can be also described in terms of colored fans.

The curves of horospherical varieties are also well-known (the intersecting ones are isomorphic to $\mathbb{P}^1$) and the intersection numbers of divisors with curves can be also described in terms of colored fans [Bri93].

3.3 Classification of projective horospherical varieties in terms of moment polytopes

The latter section permits to give a classification of projective horospherical varieties in terms of moment polytopes.

In the rest of the section, we fix a projective G/H-embedding X and a $\mathbb{Q}$-divisor $D = \sum_{i=1}^r d_i X_i + \sum_{\alpha\in\mathcal{R}} d_\alpha D_\alpha$ of X. And we suppose that D is $\mathbb{Q}$-Cartier and ample.

We denote by $X(P)$ the group of characters of P and by $X(P)^+$ the subset of dominant characters of P (and $X(P)_\mathbb{Q} = X(P)\otimes_\mathbb{Z}\mathbb{Q}$, $X(P)^+_\mathbb{Q} = X(P)^+\otimes_\mathbb{Z}\mathbb{Q}$). For any $\alpha\in\mathcal{R}$, the wall $W_{\alpha,P}$ is the intersection of $X(P)^+_\mathbb{Q}$ with the hyperplane defined by $\langle\cdot,\alpha^\vee\rangle = 0$.

Proposition 3.12. *[Pas15, Corollary 2.8]*

1. *The pseudo-moment polytope $\tilde{Q}_D$ of (X,D) is of maximal dimension in $M_\mathbb{Q}$.*
2. *The moment polytope Q_D is contained in the dominant chamber $X(P)^+_\mathbb{Q}$ and it is not contained in any wall $W_{\alpha,P}$ for $\alpha\in\mathcal{R}$.*
3. *There is a bijection between faces of Q_D (or $\tilde{Q}_D$) and G-orbits of X (preserving the respective orders). In particular, the G-stable primes divisors X_i are in bijection with the facets of Q_D that are not contained in any wall $W_{\alpha,P}$ for $\alpha\in\mathcal{R}$ (the bijection maps X_i to the facet of $\tilde{Q}_D$ defined by $\langle m,x_i\rangle = -d_i$).*
4. *The divisor D can be computed from the pair $(Q,\tilde{Q})$ as follows: the coefficients d_α with $\alpha\in\mathcal{R}$ are given by the translation vector in $X(P)_\mathbb{Q}$ that maps $\tilde{Q}$ to Q; and for any $i\in\{1,\dots,r\}$, the coefficient d_i is given by $-\langle m,x_i\rangle$ for any element $m\in M_\mathbb{Q}$ in the facet of $\tilde{Q}$ associated to X_i.*

Definition 3.13. Let Q be a polytope in $X(P)^+_\mathbb{Q}$ (not necessarily a lattice polytope). We say that Q is a **G/H-polytope**, if its direction is $M_\mathbb{Q}$ (i.e., if the subvector space of $X(P)_\mathbb{Q}$, spanned by the vectors $x-y$ with x and y in Q, is $M_\mathbb{Q}$) and if it is contained in no wall $W_{\alpha,P}$ with $\alpha\in\mathcal{R}$.

Let Q and Q' be two G/H-polytopes in $X(P)^+_\mathbb{Q}$. Consider any polytopes $\tilde{Q}$ and $\tilde{Q}'$ in $M_\mathbb{Q}$ obtained by translations from Q and Q' respectively. We say that Q and Q' are *equivalent G/H-polytopes* if the following conditions are satisfied.

1. There exist an integer j and $2j$ affine half-spaces $\mathcal{H}_1^+,\dots,\mathcal{H}_j^+$ and $\mathcal{H'}_1^+,\dots,\mathcal{H'}_j^+$ of $M_\mathbb{Q}$ (respectively delimited by the affine hyperplanes $\mathcal{H}_1,\dots,\mathcal{H}_j$ and $\mathcal{H'}_1,\dots,\mathcal{H'}_j$) such that $\tilde{Q}$ is the intersection of the $\mathcal{H}_i^+$, $\tilde{Q}'$ is the intersection of the $\mathcal{H'}_i^+$, and for all $i\in\{1,\dots,j\}$, $\mathcal{H}_i^+$ is the image of $\mathcal{H'}_i^+$ by a translation.
2. With the notation of the previous item, for all subsets J of $\{1,\dots,j\}$, the intersections $\cap_{i\in J}\mathcal{H}_i\cap\tilde{Q}$ and $\cap_{i\in J}\mathcal{H'}_i\cap\tilde{Q}'$ have the same dimension.
3. Q and Q' intersect exactly the same walls $W_{\alpha,P}$ of $X(P)^+_\mathbb{Q}$ (with $\alpha\in\mathcal{R}$).

Proposition 3.14. *[Pas15, Proposition 2.10] The map from (isomorphism classes of) projective G/H-embeddings to the set of equivalence classes of G/H-polytopes that maps X to the class of the moment polytope of (X,D), where D is any ample $\mathbb{Q}$-Cartier B-stable $\mathbb{Q}$-divisor, is a well-defined bijection.*

Since isomorphism classes of horospherical homogeneous G-spaces are in bijection with pairs (P,M) where P is a parabolic subgroup of G containing B and M is a sublattice of $X(P)$ (see [Pas15, Proposition 2.4]), we can give the following alternative classification.

Definition 3.15. A **moment quadruple** is a quadruple $(P,M,Q,\tilde{Q})$ where P is a parabolic subgroup of G containing B, M is a sublattice of $X(P)$, Q is a polytope in $X(P)^+_\mathbb{Q}$ and $\tilde{Q}$ is a polytope in $M_\mathbb{Q}$ that satisfies the three following conditions.

1. There exists (a unique) $\varpi\in\mathfrak{X}(P)_\mathbb{Q}$ such that $Q = \varpi+\tilde{Q}$.
2. The polytope $\tilde{Q}$ is of maximal dimension in $M_\mathbb{Q}$ (i.e., its interior in $M_\mathbb{Q}$ is not empty).
3. The polytope Q is not contained in any wall $W_{\alpha,P}$ for $\alpha\in\mathcal{R}$.

Definition 3.16. A **polarized horospherical variety** (respectively G/H-embedding) is a pair (X,D) such that X is a projective horospherical variety (respectively G/H-embedding) and D is an ample $\mathbb{Q}$-Cartier B-stable $\mathbb{Q}$-divisor. We say that (X,D) is isomorphic to (X',D') if there is an isomorphism from X to X' of embeddings of the same homogeneous space such that D gets identified with D' under this isomorphism.

Corollary 3.17. *The map from the set of isomorphism classes of polarized projective horospherical G-varieties to the set of classes of moment quadruples, that maps the polarized projective horospherical G-varieties (X,D) to the moment quadruples $(P,M,Q_D,\tilde{Q}_D)$ is a bijection.*

Morphisms between projective horospherical varieties can be also described in terms of moment polytopes (see for example [Pas15, Section 2.4] or [Pas17, Section 2.4]).

The curves of horospherical varieties are also well-known and the intersection numbers of $\mathbb{Q}$-Cartier divisors with curves can be also described in terms of moment polytopes (see for example [Pas15, Section 2.5] or [Pas17, Section 2.5]).

4 MMP for horospherical varieties via moment polytopes

In this section, we summarize the result of [Pas15] that describes the MMP for horospherical varieties by studying certain continuous changes of moment polytopes of polarized horospherical varieties. Note that this result can be generalized to Log MMP [Pas17].

4.1 A one-parameter family of polytopes associated to a polarized horospherical variety

Let X be a projective horospherical variety. We suppose that X is $\mathbb{Q}$-Gorenstein, i.e. K_X is $\mathbb{Q}$-Cartier.

The idea is to choose an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D of X and change continuously D by using K_X, in order to obtain a not ample but globally generated divisor D'. The ample divisor D gives an embedding from X to a projective space, while D' gives a map, from X to a projective space, that is not an embedding. This defines a contraction from X to X', where X' is the image of the latter map.

Concretely, we consider the divisor $D + \epsilon K_X$ for any rational number ϵ small enough. In terms of moment polytopes we consider the family of moment (and pseudo-moment) polytopes $Q_{D+\epsilon K_X}$ (and $\tilde{Q}_{D+\epsilon K_X}$) for ϵ small enough, that we extend to a family for any non-negative rational number ϵ. And then we study this family to deduce an MMP for X.

Let $D = \sum_{i=1}^{r} d_i X_i + \sum_{\alpha\in\mathcal{R}} d_\alpha D_\alpha$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor of X. Recall that an anti-canonical divisor of X is $-K_X = \sum_{i=1}^{k} X_i + \sum_{\alpha\in\mathcal{R}} a_\alpha D_\alpha$ (see Theorem 3.11).

We define **for any $\epsilon \in \mathbb{Q}_{\geq 0}$, the pseudo-moment and moment polytopes** $\tilde{Q}^\epsilon := \{x \in M_\mathbb{Q} \mid Ax \geq \tilde{B} + \epsilon\tilde{C}\}$ **and** $Q^\epsilon := v^\epsilon + \tilde{Q}^\epsilon$ where the matrices A, $\tilde{B}$, $\tilde{C}$ and the vector v^ϵ are defined below.

Recall that $x_1, \dots, x_k$ denote the primitive elements of N associated to the G-stable prime divisors X_i of X. Choosing an order in $\mathcal{R}$ we denote by $\alpha_1, \dots, \alpha_l$ its elements. We fix a basis $\mathcal{B}$ of M and we denote by $\mathcal{B}^\vee$ the dual basis in N. The rank of M is denoted by n.

Now define $A \in \mathcal{M}_{k+l,n}(\mathbb{Q})$ whose first k rows are the coordinates of the vectors x_i in the basis $\mathcal{B}^\vee$ with $i \in \{1, \dots, k\}$ and whose last l rows are the coordinates of the vectors $\alpha^\vee_{j|M}$ in $\mathcal{B}^\vee$ with $j \in \{1, \dots, l\}$.

Let $\tilde{B}$ be the column matrix such that the pseudo-moment polytope of D is defined by $\{x \in M_\mathbb{Q} \mid Ax \geq \tilde{B}\}$. In fact, $\tilde{B}$ is the column matrix associated to the vector $(-d_1, \dots, -d_k, -d_{\alpha_1}, \dots, -d_{\alpha_l})$.

Similarly, the column matrix $\tilde{C}$ corresponds to the vector $(1, \dots, 1, a_{\alpha_1}, \dots, a_{\alpha_l})$.

Finally, define $v^\epsilon := \sum_{\alpha\in\mathcal{R}} (d_\alpha - \epsilon a_\alpha)\varpi_\alpha$ (which is not necessarily in $M_\mathbb{Q}$).

Note that, if $M_\mathbb{Q} = X(P)_\mathbb{Q}$, we can also write the moment polytopes as follows $Q^\epsilon := \{x \in M_\mathbb{Q} \mid Ax \geq B + \epsilon C\}$ where B and C are respectively the column matrices associated to the vectors $(-d_1 + \langle x_1, \sum_{\alpha\in\mathcal{R}} d_\alpha\varpi_\alpha\rangle, \dots, -d_k + \langle x_r, \sum_{\alpha\in\mathcal{R}} d_\alpha\varpi_\alpha\rangle, 0, \dots, 0)$ and $(1 - \langle x_1, \sum_{\alpha\in\mathcal{R}} a_\alpha\varpi_\alpha\rangle, \dots, 1 - \langle x_k, \sum_{\alpha\in\mathcal{R}} a_\alpha\varpi_\alpha\rangle, 0, \dots, 0)$. Moreover, even if $M_\mathbb{Q} \neq X(P)_\mathbb{Q}$, it is easy to see that the l last inequalities defining $\tilde{Q}^\epsilon$ are equivalent to the fact that Q^ϵ is in $X^+(P)_\mathbb{Q}$.

4.2 The MMP described by the one-parameter family of polytopes: examples

To explain how to study the one-parameter family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ in order to describe the MMP for X, we give several examples. Note that the described MMP only depends on the choice of the ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor X.

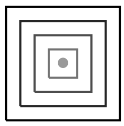
Example 4.1. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. Then $G = (\mathbb{C}^*)^2$ acts on X by $(z_1, z_2) \cdot ([a, b], [c, d]) = ([a, z_1 b], [c, z_2 d])$. Then X is toric variety with four G-stable divisors: X_1 $(b = 0)$, X_2 $(d = 0)$, X_3 $(a = 0)$ and X_4 $(c = 0)$, so that $x_1 = (1, 0)$, $x_2 = (0, 1)$, $x_3 = (-1, 0)$ and $x_4 = (0, -1)$ in the canonical basis of N (dual to the canonical basis of $M = \mathcal{X}((\mathbb{C}^*)^2)$).

Every divisor of X is linearly equivalent to some $D = d_3X_3 + d_4X_4$, because if f_m is a non-zero rational function of X of weight $m \in M$, we have $\operatorname{div}(f_m) = \sum_{i=1}^k \langle m, x_i\rangle X_i = m_1X_1 + m_2X_2 - m_1X_3 - m_2X_4$. Moreover such a $\mathbb{Q}$-divisor D is $\mathbb{Q}$-Cartier. It is globally generated (resp. ample) if and only if d_3 and d_4 are non-negative (resp. positive). In that case, the moment polytope Q_D (which equals the pseudo-moment polytope for toric varieties) is the rectangle of vertices $(0, 0)$, $(d_3, 0)$, (d_3, d_4) and $(0, d_4)$.

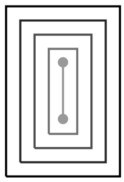
Here $-K_X = X_1 - X_2 - X_3 - X_4$, so that $D + \epsilon K_X = -\epsilon X_1 - \epsilon X_2 + (d_3 - \epsilon)X_3 + (d_4 - \epsilon)X_4$. For ϵ small enough, the moment polytope $Q_{D+\epsilon K_X}$ is the rectangle of vertices (ϵ, ϵ), $(d_3 - \epsilon, \epsilon)$, $(d_3 - \epsilon, d_4 - \epsilon)$ and $(0, d_4 - \epsilon)$.

We distinguish three cases, with positive d_3 and d_4:

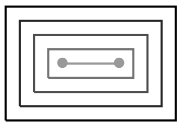
- $d_3 = d_4$: the family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ consists of rectangles for $\epsilon \in [0, d_3[$, a point for $\epsilon = d_3$ and the empty set for $\epsilon > d_3$; the MMP described by this family consists of the fibration from X to the point.



- $d_3 < d_4$: the family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ consists of rectangles for $\epsilon \in [0, d_3[$, an interval for $\epsilon = d_3$ and the empty set for $\epsilon > d_3$; the MMP described by this family consists of a fibration from X to $\mathbb{P}^1$ (projection on the second factor of $\mathbb{P}^1 \times \mathbb{P}^1$).



- $d_3 > d_4$: similarily, the MMP described by the family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ consists of a fibration from X to $\mathbb{P}^1$ (projection on the first factor of $\mathbb{P}^1 \times \mathbb{P}^1$).



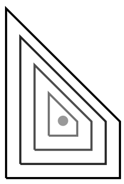
Example 4.2. Let X be $\mathbb{P}^2$ blown-up at a point. It is also a toric variety of dimension 2, with four $(\mathbb{C}^*)^2$-stable divisors X_1, X_2, X_3 and X_4 (the exceptional divisor of the blow-up), so that $x_1 = (1, 0)$, $x_2 = (0, 1)$, $x_3 = (-1, -1)$ and $x_4 = (0, -1)$.

Every divisor of X is still linearly equivalent to some $D = d_3X_3 + d_4X_4$. Moreover such a $\mathbb{Q}$-divisor D is $\mathbb{Q}$-Cartier. It is globally generated (resp. ample) if and only if d_3 and $d_3 - d_4$ are non-negative (resp. positive). And the moment polytope Q_D is the quadrilateral of vertices $(0, 0)$, $(d_4, 0)$, $(d_4, d_3 - d_4)$ and $(0, d_3)$.

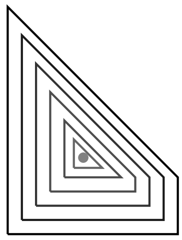
Here also $-K_X = X_1 - X_2 - X_3 - X_4$, so that $D + \epsilon K_X = -\epsilon X_1 - \epsilon X_2 + (d_3 - \epsilon)X_3 + (d_4 - \epsilon)X_4$. For ϵ small enough, the moment polytope $Q_{D+\epsilon K_X}$ is the quadrilateral of vertices (ϵ, ϵ), $(d_4 - \epsilon, \epsilon)$, $(d_4 - \epsilon, d_3 - d_4)$ and $(\epsilon, d_3 - 2\epsilon)$.

We distinguish three cases, with positive d_3 and $d_3 - d_4$:

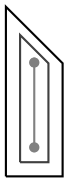
- $2d_3 = 3d_4$: the family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ consists of quadrilaterals for $\epsilon \in [0, \frac{d_3}{3}[$, a point for $\epsilon = \frac{d_3}{3}$ and the empty set for $\epsilon > \frac{d_3}{3}$; the MMP described by this family consists of the fibration of X to the point.



- $2d_3 < 3d_4$: the family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ consists of quadrilaterals for $\epsilon \in [0, d_3 - d_4[$, a triangle for $\epsilon \in [d_3 - d_4, \frac{d_4}{2}[$ and the empty set for $\epsilon > \frac{d_4}{2}$; the MMP described by this family consists of the bow-up map $X \longrightarrow \mathbb{P}^2$ (which is a divisorial contraction), and the fibration from $\mathbb{P}^2$ to the point.



- $2d_3 > 3d_4$: the family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ consists of quadrilaterals for $\epsilon \in [0, \frac{d_4}{2}[$, an interval for $\epsilon = \frac{d_4}{2}$ and the empty set for $\epsilon > \frac{d_4}{2}$; the MMP described by the family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ consists of a fibration from X to $\mathbb{P}^1$.



Example 4.3. We now consider a 3-dimensional toric variety X such that the moment polytope of an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D of X is the first polytope of Figure 4.3. Here $G = (\mathbb{C}^*)^3$ and the variety X has six G-stable prime divisors $X_1, \dots, X_6$ respectively with associated primitive elements of N: $x_1 = (0,0,1)$, $x_2 = (2,0,-1)$, $x_3 = (0,2,-1)$, $x_4 = (-2,0,-1)$, $x_5 = (0,-2,-1)$ and $x_6 = (-1,-1,-2)$. And $D = X_1 + 4X_2 + 3X_3 - 4X_4 - 3X_5 + 5X_6$.

Then for any $\epsilon \in \mathbb{Q}_{\geq 0}$, we have $Q^\epsilon := \{x \in \mathbb{Q}^3 \mid Ax \geq B + \epsilon C\}$ where

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & -1 \\ 0 & 2 & -1 \\ -2 & 0 & -1 \\ 0 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix}, \; B = \begin{pmatrix} -1 \\ -4 \\ -3 \\ 4 \\ 3 \\ -5 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

We draw some polytopes of the family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$, as ϵ grows, in Figure 4.3. We observe 5 equivalent classes of polytopes: the first one (with 8 vertices and 6 facets, for $\epsilon \in [0, \frac{1}{2}[$) corresponds to the variety X, the second one (with 7 vertices and 6 facets, for $\epsilon = \frac{1}{2}$) corresponds to a variety Y that is not $\mathbb{Q}$-Gorenstein, the third one (with 8 vertices and 6 facets, for $\epsilon \in]\frac{1}{2}, \frac{3}{2}[$) corresponds to a $\mathbb{Q}$-Gorenstein variety X^+, the fourth one (with 6 vertices and 5 facets, for $\epsilon \in [\frac{3}{2}, 2[$) corresponds to a $\mathbb{Q}$-Gorenstein variety Z and the last one (an interval, for $\epsilon = 2$) corresponds to $\mathbb{P}^1$.

The MMP for X, described by the family $(Q^\epsilon)_{\epsilon\in[0,2]}$ consists of a flip, a divisorial contraction and a fibration.

Example 4.4. Let $\bar{X}$ be the projectivization of the cylinder with $\mathrm{SO}_3(\mathbb{C}) \simeq \mathrm{PSL}_2(\mathbb{C})$ acting on X as in Example 1.11. Then $G = \mathrm{SL}_2(\mathbb{C})$ also acts on $\bar{X}$ (its center acting trivially). Let T be the maximal torus of G consisting of diagonal matrices, and B be the Borel subgroup of G consisting of upper triangular matrices. We denote by α the unique simple root of (G, B, T) and by ϖ the unique fundamental weight (G, B, T).

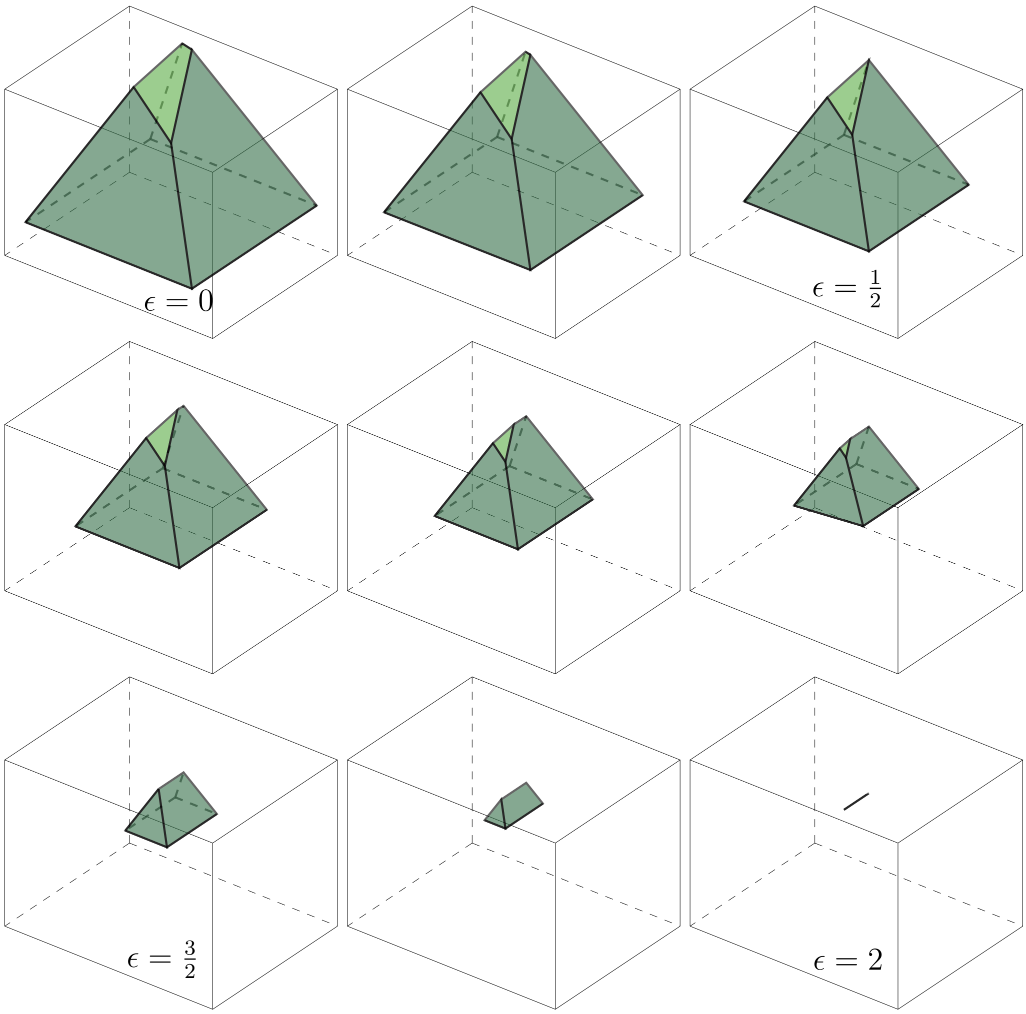
Then G acts on $\bar{X}$ with an open orbit isomorphic to G/H where H is the kernel of the character 2ϖ of B. In particular $\bar{X}$ is a horospherical variety. The lattice M is the sublattice $\mathbb{Z}(2\varpi) \simeq \mathbb{Z}$ of $X(B) = \mathbb{Z}\varpi$. The homogeneous space G/H has one color associated to the unique simple root α of $\mathrm{SL}_2(\mathbb{C})$, and $\alpha^\vee_M = 2$. Note also that $X(B)^+_{\mathbb{Q}}$ is the half-line $\mathbb{Q}_{\geq 0}\varpi$.

Moreover there are exactly two projective G/H-embeddings: $\bar{X}$ that corresponds for example to the moment polytope $[0, 2\varpi]$ (note that $V(0) \oplus V(2\varpi) = \mathbb{C} \oplus S^2\mathbb{C}^2 \simeq \mathbb{C}^4$), and $\tilde{X}$ that corresponds for example to the moment polytope $[2\varpi, 4\varpi]$.

The MMP for $\bar{X}$ only consists of a fibration to a point (as we saw in Example 1.11): the family $(Q^\epsilon)_{\epsilon\in[0,\frac{1}{2}]}$ is the family of intervals $[0, 2(1-2\epsilon)\varpi]$. And the MMP for $\tilde{X}$ only consists of a fibration to $\mathrm{SL}_2/B \simeq \mathbb{P}^1$: the family $(Q^\epsilon)_{\epsilon\in[0,\frac{1}{2}]}$ is the family of intervals $[2\varpi, 4(1-\epsilon)\varpi]$.

Note that there is a contraction from $\tilde{X}$ to $\bar{X}$, but it contracts zero curves, in particular it cannot appear in the classical MMP.

Figure 2: Some polytopes of the family $(Q^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$ in Example 4.3



Example 4.5. Let $G = \mathrm{SL}_4(\mathbb{C})$. Let B be the Borel subgroup of G consisting of upper triangular matrices, and T the maximal torus consisting of diagonal matrices. Denote by α_1, α_2 and α_3 the three simple roots of (G, B, T) with Bourbaki's notation, and by ϖ_1, ϖ_2 and ϖ_3 the corresponding fundamental weights.

Let P be the minimal parabolic subgroup containing B and associated to the simple root α_3, so that $\mathcal{R} = \{\alpha_1, \alpha_2\}$. And let H be the kernel of the character $\varpi_1 + 2\varpi_2 : P \longrightarrow \mathbb{C}^*$. In fact

$$H = \{ \begin{pmatrix} t^2 & * & * & * \\ 0 & t^{-3} & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \in \mathrm{SL}_4(\mathbb{C}), \mid\ t \in \mathbb{C}^* \}.$$

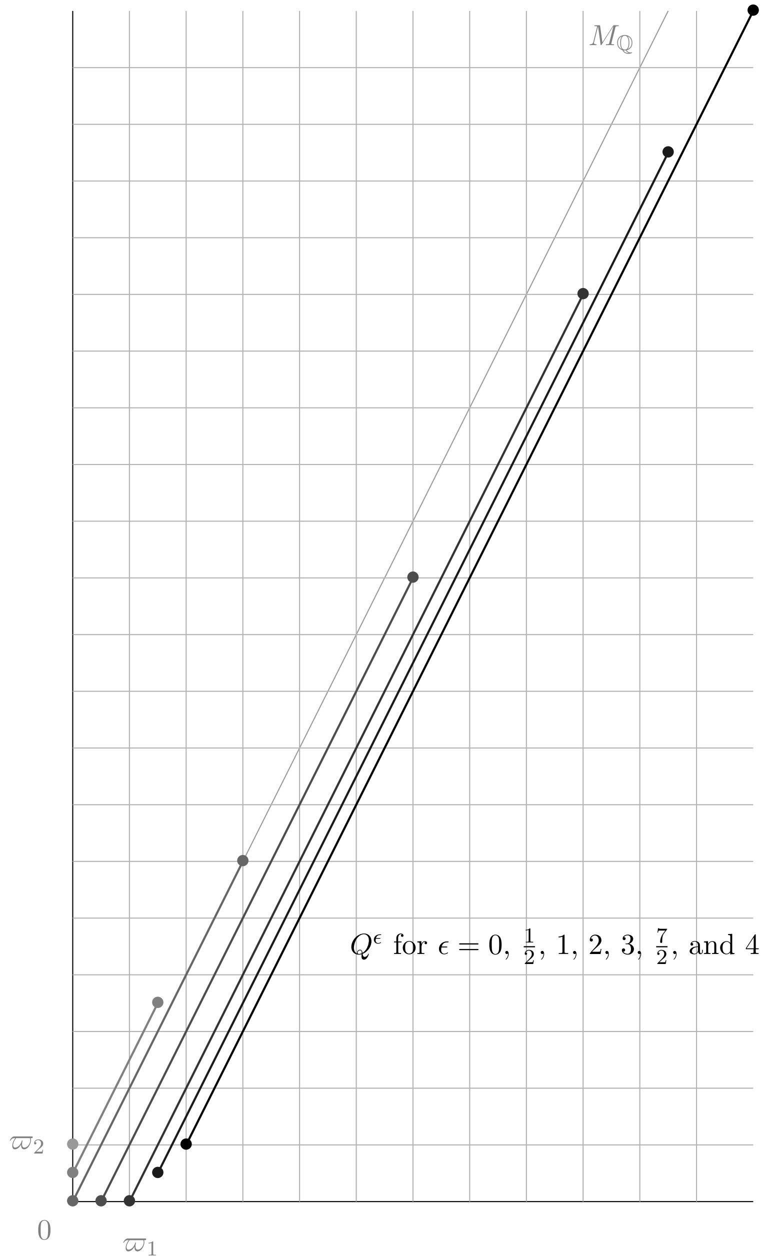
Then $M = \mathbb{Z}(\varpi_1 + 2\varpi_2) \simeq \mathbb{Z}$ Moreover, $X(P) = \mathbb{Z}\varpi_1 \oplus \mathbb{Z}\varpi_2$ and $X(P)^+ = \mathbb{N}\varpi_1 \oplus \mathbb{N}\varpi_2$. Note also that $n = 1$, G/H is 6-dimensional and $M_{\mathbb{Q}} \neq X(P)_{\mathbb{Q}}$. Consider the unique projective G/H-embedding X without color. It has two G-stable prime divisors X_1 and X_2, with $x_1 = 1 (= \varpi_1 + 2\varpi_2)$ and $x_2 = -1$, two other B-stable prime divisors D_{α_1} and D_{α_2} whose images respectively $\alpha_{1M}^\vee = 1$ and $\alpha_{2M}^\vee = 2$. We can also compute that $a_{\alpha_1} = 2$ and $a_{\alpha_2} = 3$.

Let $D = X_1 + 9X_2 + 3D_{\alpha_1} + 3D_{\alpha_2}$.

Then $A = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \end{pmatrix}$ and $\tilde{B} = \begin{pmatrix} -1 \\ -9 \\ -3 \\ -3 \end{pmatrix}$. Moreover $-K_X = X_1 + X_2 + 2D_{\alpha_1} + 3D_{\alpha_2}$, so that $\tilde{C} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ and $v^\epsilon = (3 - 2\epsilon)\varpi_{\alpha_1} + (3 - 3\epsilon)\varpi_{\alpha_2}$.

We draw some polytopes (intervals) of the family $(Q^\epsilon)_{\epsilon \in \mathbb{Q}_{\geq 0}}$ in Figure 4.5. The MMP for X, described by the family $(Q^\epsilon)_{\epsilon \in [0,4]}$ consists of a divisorial contraction (adding a color), a flip (exchanging colors) and a fibration to the Grassmannian $\mathrm{SL}_4(\mathbb{C})/P(\varpi_2)$ of planes in $\mathbb{C}^4$.

Figure 3: Some polytopes of the family $(Q^{\epsilon})_{\epsilon \in \mathbb{Q}_{\geq 0}}$ in Examples 4.5



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