SLODOWY SLICES

1. Jacobson-Morozov theorem

This theorem is a key tool to study nilpotent elements and nilpotent orbits in a semisimple Lie algebra.

Let G be a connected semisimple algebraic group over $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra, and $e \in \mathfrak{g}$ be a nilpotent element. Recall that e is called nilpotent if one of the following equivalent conditions hold:

(1) e is represented by a nilpotent operator in some faithful finite dimensional representation of $\mathfrak{g}$.
(2) e is represented by a nilpotent operator in every finite dimensional representation of $\mathfrak{g}$.
(3) We have $f(e) = f(0)$ for any G-invariant polynomial f on $\mathfrak{g}$.
(4) $0 \in \overline{Ge}$.

Theorem 1.1. *Every nilpotent element $e \in \mathfrak{g}$ can be included into an $\mathfrak{sl}_2$-triple: there are elements $h, f \in \mathfrak{g}$ with $[h, e] = 2e, [h, f] = -2f, [e, f] = h$.*

In fact, we have the following results of Dynkin and Kostant which make the theorem more precise.

Theorem 1.2. *Let $(e, h, f), (e, h', f')$ be two $\mathfrak{sl}_2$-triples. Then there is an element $g \in G$ centralizing e such that $gh = h', gf = f'$.*

Theorem 1.3. *Let $(e, h, f), (e', h, f')$ be two $\mathfrak{sl}_2$-triples. Then there is an element $g \in G$ centralizing h such that $ge = e', gf = f'$.*

Problem 1. Prove Theorem 1.1 in the case of $\mathfrak{g} = \mathfrak{sl}_n$.

Problem 2. Prove Theorem 1.1 for the general $\mathfrak{g}$. You may use the following strategy:

1) Check that $x \in \mathfrak{g}$ lies in the image of $\operatorname{ad} e$ if and only if x is orthogonal (w.r.t. the Killing form) to the centralizer of e.

2) Prove Theorem 1.1 in the case when the centralizer of e consists of nilpotent elements.

3) Prove Theorem 1.1 in the general case.

Let $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}(i)$ be the grading by eigenvalues of $\operatorname{ad} h$.

Problem 2'. For $i > 0$, let x_i be an element of $\mathfrak{g}(i)$ such that $[e, x_i] = 0$. Set $x := \sum_{i>0} x_i$. Prove that there is an element g in the unipotent radical of $Z_G(e)$ that maps h to $h + x$. Deduce Theorem 1.2.

Problem 2". Show that $Z_G(h)$ acts on $\mathfrak{g}(2)$ with an open orbit. Deduce Theorem 1.3.

The next three problems concern applications of the theorems.

Problem 3. Show that the nilpotent orbits in $\mathfrak{g}$ are in one-to-one correspondence with the G-conjugacy classes of Lie algebra homomorphisms $\mathfrak{sl}_2 \to \mathfrak{g}$.

Problem 4. Show that the number of nilpotent orbits in $\mathfrak{g}$ is finite.

Problem 5. Describe the nilpotent orbits for $\mathrm{O}(n)$ and $\mathrm{Sp}(2n)$. How is the case of $\mathrm{SO}(n)$ different from that of $\mathrm{O}(n)$?

2. Slodowy slices

Yet one more application of the Jacobson-Morozov theorem is a construction of slices to nilpotent orbits. Let e, h, f be an $\mathfrak{sl}_2$-triple. Set $S := e + \ker \operatorname{ad} f$. This is a so called *Slodowy slice*.

Problem 6. Show that the intersection of S and Ge at e is transversal.

Define the action of $\mathbb{C}^\times$ on $\mathfrak{g}$ by $t \cdot x = t^{2-i}x$ for $x \in \mathfrak{g}(i)$.

Problem 7. Show that this action preserves S and contracts it to the point e.

Problem 8. Show that $S \cap Ge = \{e\}$. Moreover, show that $T_sS + T_sGs = \mathfrak{g}$ for any $s \in S$.