1. JACOBSON-MOROZOV THEOREM

This theorem is a key tool to study nilpotent elements and nilpotent orbits in a semisimple Lie algebra.

Let $G$ be a connected semisimple algebraic group over $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra, and $e \in \mathfrak{g}$ be a nilpotent element. Recall that $e$ is called nilpotent if one of the following equivalent conditions hold:

1. $e$ is represented by a nilpotent operator in some faithful finite dimensional representation of $\mathfrak{g}$.
2. $e$ is represented by a nilpotent operator in every finite dimensional representation of $\mathfrak{g}$.
3. We have $f(e) = f(0)$ for any $G$-invariant polynomial $f$ on $\mathfrak{g}$.
4. $0 \in \mathfrak{g} e$.

**Theorem 1.1.** Every nilpotent element $e \in \mathfrak{g}$ can be included into an $\mathfrak{sl}_2$-triple: there are elements $h, f \in \mathfrak{g}$ with $[h, e] = 2e, [h, f] = -2f, [e, f] = h$.

In fact, we have the following results of Dynkin and Kostant which make the theorem more precise.

**Theorem 1.2.** Let $(e, h, f), (e', h', f')$ be two $\mathfrak{sl}_2$-triples. Then there is an element $g \in G$ centralizing $e$ such that $gh = h', gf = f'$.

**Theorem 1.3.** Let $(e, h, f), (e', h, f')$ be two $\mathfrak{sl}_2$-triples. Then there is an element $g \in G$ centralizing $h$ such that $ge = e', gf = f'$.

**Problem 1.** Prove Theorem 1.1 in the case of $\mathfrak{g} = \mathfrak{sl}_n$.

**Problem 2.** Prove Theorem 1.1 for the general $\mathfrak{g}$. You may use the following strategy:

1) Check that $x \in \mathfrak{g}$ lies in the image of $\text{ad} e$ if and only if $x$ is orthogonal (w.r.t. the Killing form) to the centralizer of $e$.
2) Prove Theorem 1.1 in the case when the centralizer of $e$ consists of nilpotent elements.
3) Prove Theorem 1.1 in the general case.

Let $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ be the grading by eigenvalues of $\text{ad} h$.

**Problem 2'.** For $i > 0$, let $x_i$ be an element of $\mathfrak{g}(i)$ such that $[e, x_i] = 0$. Set $x := \sum_{i > 0} x_i$. Prove that there is an element $g$ in the unipotent radical of $Z_G(e)$ that maps $h$ to $h + x$.

Deduce Theorem 1.2.

**Problem 2''.** Show that $Z_G(h)$ acts on $\mathfrak{g}(2)$ with an open orbit. Deduce Theorem 1.3.

The next three problems concern applications of the theorems.

**Problem 3.** Show that the nilpotent orbits in $\mathfrak{g}$ are in one-to-one correspondence with the $G$-conjugacy classes of Lie algebra homomorphisms $\mathfrak{sl}_2 \rightarrow \mathfrak{g}$.

**Problem 4.** Show that the number of nilpotent orbits in $\mathfrak{g}$ is finite.

**Problem 5.** Describe the nilpotent orbits for $O(n)$ and $\text{Sp}(2n)$. How is the case of $\text{SO}(n)$ different from that of $O(n)$?
2. Slodowy slices

Yet one more application of the Jacobson-Morozov theorem is a construction of slices to nilpotent orbits. Let $e, h, f$ be an $\mathfrak{sl}_2$-triple. Set $S := e + \ker \text{ad} f$. This is a so called Slodowy slice.

**Problem 6.** Show that the intersection of $S$ and $Ge$ at $e$ is transversal.

Define the action of $\mathbb{C}^\times$ on $\mathfrak{g}$ by $t \cdot x = t^{2i}x$ for $x \in \mathfrak{g}(i)$.

**Problem 7.** Show that this action preserves $S$ and contracts it to the point $e$.

**Problem 8.** Show that $S \cap Ge = \{e\}$. Moreover, show that $T_sS + T_sGs = \mathfrak{g}$ for any $s \in S$. 