Lec 1: Nilpotent orbits

1) Nilp. orbits & $\mathfrak{sl}_2$-triples

2) Appl'ns of $\mathfrak{sl}_2$-triples

3) Sympl'c sing's.

1.1) Nilp. el'ts & orbits: conn'd s/simple $G(/\mathbb{C}) \curvearrowright \mathfrak{g} = Lie(G)$

Thm-Def'n: $x \in \mathfrak{g}$ is nilp't if follow. equiv't cond'ns hold:

(1) $0 \in \overline{Gx}$ (Zariski topology)

(2) $\varphi(x)$ is nilp. $\forall$ rep'n $\varphi: \mathfrak{g} \to \mathfrak{gl}(V)$ (fin. dim'l)

(3) –·–·–·–· for some faith'l –·–·–·–·–·–

E.g. if $\mathfrak{g}$ is classical, then nilp't el't = nilp. matrix in $\mathfrak{g}$

Def'n: nilp. orbit: $\mathbb{O} := Gx$, $x \in \mathfrak{g}$ nilp

1.2) $\mathfrak{sl}_2$-triples: Def'n: $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ is (e,h,f) w. $[h,e]=2e, [h,f]=-2f, [e,$

$= h \leftrightarrow$ homom. $\varphi: \mathfrak{sl}_2 \to \mathfrak{g}$ $\quad \begin{pmatrix}0&1\\0&0\end{pmatrix} \begin{pmatrix}1&0\\0&-1\end{pmatrix} \begin{pmatrix}0&0\\1&0\end{pmatrix}$

Thm (Jacobson-Morozov, existence) $\forall$ nilp. $e \in \mathfrak{g}$ $\exists$ $\mathfrak{sl}_2$-triple (e,h,f)

Thm (Kostant, uniqueness) $(e,h,f), (e,h',f')$ $\mathfrak{sl}_2$-triples $\Rightarrow \exists g \in Z_G(e) \mid gh = h', gf =$

Cor: $\{\text{nilp. orbits}\} \xleftrightarrow{\sim} \{\varphi: \mathfrak{sl}_2 \to G\}/G$ $(*)$

Morally, it's easier to understand the r.h.s. $\mathfrak{sl}_2$-triples overall is a crucial tool to study various questions about nilpotent orbits

2.1) Appl'n: classif'n of nilp. orbits

- $\mathfrak{g} = \mathfrak{sl}_n$: $(*) = n$-dim'l $\mathfrak{sl}_2$-reps/iso $\Longleftrightarrow$ Young diagrams (so nilp. orbits are classified by Young diagrams – by Jordan type)
- $\mathfrak{g} = \mathfrak{so}_n$ ($\&$ $G = O_n$ - disconnected, comp-t group is $\mathbb{Z}/2\mathbb{Z}$) or $\mathfrak{g} = \mathfrak{sp}_n$ (n even

$(*) = n$-dim'l orth/sympl. $\mathfrak{sl}_2$-reps/orth or sympl. iso $\longleftarrow$ isom

if there's an iso can choose it orthog or sympl

- compare to the fact that up to conj. there's a single orthog/sympl. form on $\mathbb{C}$

So our question is: when is $\mathfrak{sl}_2$-rep orthogonal/symplectic

- V-vect. space $\Rightarrow V \oplus V^*$ have nat'l orth & sympl. form

- $\mathfrak{sl}_2$-irrep V is orthog if $\dim V$ odd, sympl. if $\dim V$ even $\Rightarrow$ self-dual

Result: V $\mathfrak{sl}_2^\vee$-rep; $\forall$ even/odd dim'l irrep appears in V w. even mult. $\Leftrightarrow$ V is orthog/sympl.

Concl'n: Nilp O_n-orbits in $\mathfrak{so}_n$ / Sp_n-orbits in $\mathfrak{sp}_n$ $\xleftrightarrow{\sim}$ Young diagr., where even/odd parts have even mult.

Not'n: $\lambda \vdash n \rightsquigarrow \mathbb{O}_\lambda \in \mathfrak{g}$

Rem: O_n-orbit $\mathbb{O}_\lambda$ splits into 2 SO_n-orbits $\Leftrightarrow$ all parts of λ are even $(\Rightarrow$ ne

Otherwise, $\mathbb{O}_\lambda$ is a single SO_n-orbit.

Gen'l Thm: # nilp. orbits in $\mathfrak{g} < \infty$

2.2) Resol'n of sing's for $\overline{\mathbb{O}}$

Not-n: $\mathfrak{g}_i := \{x \in \mathfrak{g} \mid [h,x] = ix\} \rightsquigarrow \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ Lie algebra grading

$\mathfrak{g}_{\geq 0} = \bigoplus_{i \geq 0} \mathfrak{g}_i$ parabolic subalgebra $\rightsquigarrow G_{\geq 0} \subset G$, $G_{\geq 0} \curvearrowright \mathfrak{g}_{\geq 2} \rightsquigarrow$

$$\begin{tikzcd} G \times_{G_{\geq 0}} \mathfrak{g}_{\geq 2} \arrow[d] \\ \mathfrak{g} \end{tikzcd}$$

$G \times_{G_{\geq 0}} \mathfrak{g}_{\geq 2} = G \times \mathfrak{g}_{\geq 2} / \sim$: $(g,x) \sim (gh^{-1}, hx)$, $h \in G_{\geq 0}$, $[g,x] :=$ equiv. class of (g,x)

$\tau : [g,x] \mapsto gx$ (generalized Springer morphism)

Prop: $\operatorname{im} \tau = \overline{\mathbb{O}}$ & $\tau : G \times_{G_{\geq 0}} \mathfrak{g}_{\geq 2} \to \overline{\mathbb{O}}$ is resol'n of sing's

Proof: - τ is proj've:

$$\begin{tikzcd} G \times_{G_{\geq 0}} \mathfrak{g}_{\geq 2} \arrow[r, hook] \arrow[rd, "\tau"'] & G \times_{G_{\geq 0}} \mathfrak{g} = G/G_{\geq 0} \times \mathfrak{g} \arrow[d, "pr_2"] \\ & \mathfrak{g} \end{tikzcd}$$

partial flag variety

- $\operatorname{im} \tau = \overline{\mathbb{O}} \Leftarrow \overset{e \in \mathfrak{g}_2}{\mathbb{O}} \subset \operatorname{im} \tau \subset \overline{\mathbb{O}}$

$[\mathfrak{g}_{\geq 0}, e] = \mathfrak{g}_{\geq 2}$ (by rep th. of $\mathfrak{sl}_2^\vee$) $\Rightarrow \overline{G_{\geq 0} e} = \mathfrak{g}_{\geq 2}$

- τ is birat'l $\Leftarrow \tau : G \times_{G_{\geq 0}} G_{\geq 0} e \xrightarrow{\sim} \mathbb{O} \Leftrightarrow$ $Z_{G_{\geq 0}}(e) = Z_G(e)$ (*)

single G-orbit w. stabilizer

- Lie algebras are the same $\mathfrak{z}_{\mathfrak{g}_{\geq 0}}(e) = \mathfrak{z}_{\mathfrak{g}}(e)$ - from rep'n theory of $\mathfrak{sl}_2^\vee$

$N \subset Z_G(e)$ conn. subgr'p w. $\mathrm{Lie}(N) = \mathfrak{z}_{\mathfrak{g}}(e) \cap \mathfrak{g}_{\geq 1}$ - norm. & unip

$Q := Z_G(e) \cap G_0 = Z_G(e,h) (= Z_G(e,h,f)$ reductive$)$

(*) $\Leftarrow Z_G(e) = Q \ltimes N \Leftarrow Z_G(e)h = Nh$ - exer. $\square$

3.1) Sympl'c sing's: X <u>normal</u> alg. var'y, (X^{reg}, ω) sympl-c var'y

Def'n (Beauville'00) X has sympl'c sing's if $\exists (\Leftrightarrow \forall)$ resol'n of sing's

$\rho: \tilde{X} \to X$ s.t. $\rho^*(\omega)$ extends from $\rho^{-1}(X^{reg})$ to $\tilde{X}$ (may become degenerate)

Rem: sympl'c sing'y is Cohen-Macaulay, Gorenstein

3.2) Example: norm'n of $\overline{\mathbb{O}}$: $\mathbb{O} \subset \mathfrak{g}^*$ G-orbit (for any alg. gr-p G) is sympl'c.

$T_\alpha \mathbb{O} = \mathfrak{g}.\alpha$, $\omega_\alpha(x.\alpha, y.\alpha) := \langle \alpha, [x,y] \rangle$ Kirillov-Kostant form

ω is G-inv & sympl'c $\Rightarrow$ $\dim \mathbb{O}$ even

$\mathfrak{g}$ s/simple $\Rightarrow \mathfrak{g} \simeq \mathfrak{g}^* \Rightarrow$ nilp. orb. $\mathbb{O}$ is sympl'c

$\#$ nilp. orbits $< \infty \Rightarrow \operatorname{codim}_{\overline{\mathbb{O}}} \partial \mathbb{O} \geqslant 2$, $\overline{\mathbb{O}}$ may fail to be normal

$X = \operatorname{Spec} \mathbb{C}[\mathbb{O}]$ is norm-n of $\overline{\mathbb{O}}$.

Thm (Panyushev) X ~~$\operatorname{Spec} \mathbb{C}[\mathbb{O}]$~~ has sympl. sing's

Proof:

$$\begin{tikzcd}
\tilde{X} := G \times_{G_{\geqslant 0}} \mathfrak{g}_{\geqslant 2} \arrow[d, "\rho"'] \arrow[dr, "\tau"] & \\
X \cancel{\operatorname{Spec} \mathbb{C}[\mathbb{O}]} \arrow[r] & \overline{\mathbb{O}}
\end{tikzcd}$$

Enough to show ω exts from $\mathbb{O} \subset \tilde{X}$ to $\tilde{X}$

$T_{[1,x]} \tilde{X} = T_{[1,x]} G_{<0}[1,x] \oplus \mathfrak{g}_{\geqslant 2} = \mathfrak{g}_{<0} \oplus \mathfrak{g}_{\geqslant 2}$

Exer: $\exists !$ G-inv 2-form $\tilde{\omega}$ on $\tilde{X}$ w $\tilde{\omega}_{[1,x]}(y_{<0} + y_{\geqslant 2}, z_{<0} + z_{\geqslant 2}) =$
$= (x, [y_{<0}, z_{<0}]) + (y_{<0}, z_{\geqslant 2}) - (z_{<0}, y_{\geqslant 2})$. ~~Its restr-n to $\mathbb{O} = \omega$~~ $\tilde{\omega}|_{\mathbb{O}} = \omega$

$\tilde{\omega}$ is non-deg-e $\Leftrightarrow \mathfrak{g}_1 = 0$

Goal of this course: study deformations of $\mathbb{C}[\mathbb{O}]$ for nilp. orbits $\mathbb{O}$ (& some covers).