Lec 2: Around Lusztig-Spaltenstein induction

1) Ind'n for nilp. orbits

2) Poisson str's

1.1) Main result:
G conn. s/simple alg. gr'p/$\mathbb{C} \rightsquigarrow \mathfrak{g}$; $\mathfrak{l} \subset \mathfrak{g}$ Levi subalg. (= centralizer of s/simple el-t), $\mathbb{O}' \subset \mathfrak{l}$ nilp. orbit

Goal: $(\mathfrak{l}, \mathbb{O}') \rightsquigarrow$ nilp. $\mathbb{O} \subset \mathfrak{g}$

Constr'n: pick parab. $\underline{\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{n}} \Rightarrow \mathbb{O}' + \mathfrak{n} \subset$ nilp. el-ts $\Rightarrow \overline{G(\mathbb{O}'+\mathfrak{n})} \subset$ nilp. el-ts

$\mathbb{O} :=$ open orbit in $\overline{G(\mathbb{O}'+\mathfrak{n})}$

Thm: $\mathbb{O}$ is indep. of $\mathfrak{p}$ (only of $(\mathfrak{l}, \mathbb{O}')$) & $\operatorname{codim}_{\mathfrak{g}} \mathbb{O} = \operatorname{codim}_{\mathfrak{l}} \mathbb{O}'$ (1)

1.2) Generalized Springer map (main ingredient of proof)

$\overline{\mathbb{O}}'$ is L-stable $\Rightarrow \overline{\mathbb{O}}' + \mathfrak{n}$ is P-stable $\rightsquigarrow G \times_P (\overline{\mathbb{O}}'+\mathfrak{n}) \xrightarrow{\pi} \mathfrak{g}$, $[g, x] \mapsto gx$

$\dim G \times_P (\overline{\mathbb{O}}'+\mathfrak{n}) = \dim \mathfrak{g} - \dim \mathfrak{l} + \dim \mathbb{O}'$ proj-ve, $\operatorname{im} \pi = \overline{\mathbb{O}}$

$\Rightarrow$ (1) $\Leftrightarrow \pi$ is gen. finite. Term'y: $\mathbb{O}$ is birationally ind-d from $(\mathfrak{l}, \mathbb{O}')$ if π is birational

Proof of Thm: $\xi \in \mathfrak{g} \mid \mathfrak{z}_{\mathfrak{g}}(\xi) = \mathfrak{l}$ & homog. $F \in \mathbb{C}[\mathfrak{g}]^G \mid F(\xi) \neq 0 \rightsquigarrow$

$$\begin{tikzcd} G \times_P (\mathbb{C}\xi + \overline{\mathbb{O}}' + \mathfrak{n}) \arrow[r, "\tilde{\pi}"] \arrow[d, "\text{coeff. of } \xi"'] & \mathfrak{g} \arrow[d, "F"] \\ \mathbb{C} \arrow[r, "t \mapsto F(t\xi)"'] & \mathbb{C} \end{tikzcd}$$

$\tilde{\pi}_t := \tilde{\pi}|_{G \times_P (t\xi + \overline{\mathbb{O}}' + \mathfrak{n})}$ so $\tilde{\pi}_0 = \pi$

irred.

$\operatorname{im} \pi = \operatorname{im} \tilde{\pi} \cap F^{-1}(0) \Rightarrow \dim \operatorname{im} \pi = \dim \operatorname{im} \tilde{\pi} - 1 = \dim \operatorname{im} \tilde{\pi}_1$ & $\operatorname{im} \tilde{\pi} = \overline{\mathbb{C}^\times \operatorname{im} \tilde{\pi}_1}$

dilation action

$\operatorname{im} \tilde{\pi}_1 = G(\xi + \overline{\mathbb{O}}' + \mathfrak{n}) = [y \in \mathbb{O}', [\xi, \cdot] : \mathfrak{n} \xrightarrow{\sim} \mathfrak{n} \Rightarrow [\xi + y, \cdot] : \mathfrak{n} \xrightarrow{\sim} \mathfrak{n} \Rightarrow$ $\xi + \mathbb{O}' + \mathfrak{n} \subset G(\xi+y)] = \overline{G(\xi+y)}$ - indep. of $\mathfrak{p} \Rightarrow \operatorname{im} \tilde{\pi}$ is indep $\Rightarrow \operatorname{im} \pi$ is indep

$\dim \overline{\mathbb{O}} = \dim \operatorname{im} \pi = \dim \operatorname{im} \tilde{\pi}_1 = \dim G(\xi+y) = \dim G - \dim Z_G(\xi+y) = [Z_G(\xi) = L]$
$= \dim G - \dim Z_L(y) = \dim G - \dim L + \dim \mathbb{O}'$ $\square$

Rems: 1) LS ind-n is transitive.

2) $\exists$ open G-orbit $\widetilde{\mathbb{O}} \subset G \times_P (\overline{\mathbb{O}}'+\mathfrak{n})$, cover of $\mathbb{O}$: $\widetilde{\mathbb{O}} = G/H$ w. $Z_G(e)^\circ \subset H \subset Z_G(e)$, $e \in \mathbb{O}$, $\widetilde{\mathbb{O}}$ is "ind-d cover". It's indep. of $\mathfrak{p}$

3) $\xi \in \mathfrak{z}(\mathfrak{l}) \rightsquigarrow$ ind-d orbit from $(\mathfrak{l}, \mathbb{O}', \xi)$ = open in $G(\xi + \overline{\mathbb{O}}' + \mathfrak{n})$ indep. of $\mathfrak{p}$ codim = $\operatorname{codim}_{\mathfrak{l}} \mathbb{O}'$

1.3) Examples: $\mathfrak{g}=\mathfrak{sl}_n$: conj. classes of $\mathfrak{l}$ $\longleftrightarrow$ Young diagrams: $\lambda \to \mathfrak{l}_\lambda$. Orbit ind-d from $(\mathfrak{l}_\lambda, 0)$ is $\mathbb{O}_{\lambda^t}$ & $\pi: G\times_P \mathfrak{n} = T^*(G/P) \longrightarrow \overline{\mathbb{O}}_{\lambda^t}$ is birat-l.

For the other classical Lie algebras it's much more complicated

Def: $\mathbb{O}$ is (birationally) rigid if it's not induced from $(\mathfrak{l},\mathbb{O}')$, $\mathfrak{l}\neq\mathfrak{g}$ (s.t. π is birat-l.)

So rigid $\Rightarrow$ birat. rigid but not vice versa

Rem: same $\mathbb{O}$ can be ind-d from >1 rigid orbits in diff-t Levi's: $\mathbb{O}_{(3,3,2,1,1)}\subset\mathfrak{sp}_{10}$.

· $\mathbb{O}_\lambda\subset\mathfrak{so}_n/\mathfrak{sp}_n$ is birat. rigid $\Longleftrightarrow$ $\lambda_i-\lambda_{i+1}\leq 1$. $\mathbb{O}_\lambda$ is rigid $\underbrace{\text{if}}_{\text{in addition}}$ mult of every odd (for $\mathfrak{so}_n$) or even (for $\mathfrak{sp}_n$) part $\neq 2$

2.1) Poisson alg-s $\mathbb{C}[\tilde{\mathbb{O}}]$.

$(\mathfrak{l},\mathbb{O}')$ mostly interested in bir. rigid $\mathbb{O}'$ $\rightsquigarrow$ $X'=\operatorname{Spec}\mathbb{C}[\mathbb{O}']$ $\rightsquigarrow$

$$\begin{tikzcd} G\times_P(X'\times\mathfrak{n}) \arrow[rr, "\pi"] \arrow[dr, "\pi'"'] & & \overline{\mathbb{O}} \\ & X \arrow[ur, "\text{fin}"'] & \end{tikzcd}$$

birat. w. conn. fibers

normal

consider Stein decomposition

$\Rightarrow X=\operatorname{Spec}\mathbb{C}[\tilde{\mathbb{O}}]$, $\tilde{\mathbb{O}}\subset G\times_P(X'\times\mathfrak{n})$ - open G-orbit

sympl-c (form lifted from KK form on

$\mathbb{C}[\tilde{\mathbb{O}}]$ is a Poisson alg-a w. grading:

$\mathbb{C}^\times\curvearrowright\mathbb{O}',\mathfrak{n}$ (dilations) $\rightsquigarrow$ $\mathbb{C}^\times\curvearrowright G\times_P(X'\times\mathfrak{n})$ (fiberwise dilations) $\rightsquigarrow$ $\mathbb{C}^\times\curvearrowright\tilde{\mathbb{O}}$ $\rightsquigarrow$ $\mathbb{C}^\times\curvearrowright X$

Rem: $\deg\{\cdot,\cdot\}=-1$ b/c $X\to\overline{\mathbb{O}}$ is equiv.

We will study filt-d Poisson & non-commut. deformations of $\mathbb{C}[\tilde{\mathbb{O}}]$. For this, we need to underst some things about the partial resolution $G\times_P(X'\times\mathfrak{n})$. We will need to know that this variety is Poisson - meaning that the structure sheaf carries a Poisson bracket. We will also need an algebro-geometric property of $G\times_P(X'\times\mathfrak{n})$ - that it's $\mathbb{Q}$-factoria terminal - here we use $\mathbb{O}'$ is bir. rigid. We will discuss this in the ~~next~~ lecture. To construct the Poisson str-re, we'll need moment maps & Hamiltonian reduction

2.2) Moment maps & Ham. red-n

Y - Poisson var'y, $G\curvearrowright Y$ preserves $\{\cdot,\cdot\}$ $\rightsquigarrow$ $\mathfrak{g}\to\mathrm{Vect}(Y)$, $\xi\mapsto\xi_Y$

Def'n: Comoment map, $\xi\mapsto H_\xi$, is a G-equiv. lin. map $\mathfrak{g}\to\mathbb{C}[Y]$ s.t. $\{H_\xi,\bullet\}=\xi_Y$. Moment map $Y\xrightarrow{\mu}\mathfrak{g}^*$ is given by $\langle\mu(y),\xi\rangle:=H_\xi(y)$

Note that $\xi\mapsto H_\xi$ is Lie alg. homom-m so $\Rightarrow$ μ is Poisson morphism fiberwise linear functions

Ex: $G \curvearrowright Y_0$ (smooth), $Y = T^*Y_0$, $\mathrm{Vect}(Y_0) \subset \mathbb{C}[T^*Y_0]$, $H_\xi := \xi_{Y_0}$

Fact (Ham. red'n) $\alpha \in (\mathfrak{g}^*)^G \rightsquigarrow G \curvearrowright \mu^{-1}(\alpha) \subset Y$. Suppose there's quotient $\mu^{-1}(\alpha)/G$.

Then it has nat'l Poisson str-re: to take bracket of two functions on $\mu^{-1}(\alpha)/G$ we lift them to $\mu^{-1}(\alpha)$, extend to Y in some way, take bracket there, restrict to μ^{-1} and get G-invariant function, that is the bracket we want to compute + (*)

(*) If Y is symplc & $G \curvearrowright \mu^{-1}(\alpha)$ free $\Rightarrow \mu^{-1}(\alpha)/G$ is symplc

2.3) Poisson str-re on $G \times_P (X' \times \mathfrak{n})$

$Y = T^*(G/N) \times X' = (G \times_N \mathfrak{p}) \times X' \curvearrowleft G \times L$ w. moment maps $\mu_G : Y \to \mathfrak{g}$, $\mu_L : Y \to \mathfrak{l}$

$\mathrm{pr}: \mathfrak{p} \to \mathfrak{l}$, $\varphi : X' \to \mathfrak{l}$ via $X' \twoheadrightarrow \bar{\mathbb{O}}' \hookrightarrow \mathfrak{l}$, φ = moment map for $L \curvearrowright X'$

$\mu_G([g,x], x') = g.x$, $\mu_L([g,x],x') = -p_{\mathfrak{l}}(x) + \varphi(x') \Rightarrow$

$$\mu_L^{-1}(0) = G \times_N (X' + \mathfrak{n}) \Rightarrow \mu_L^{-1}(0)/L = G \times_P (X' \times \mathfrak{n})$$

Poisson

$$\begin{tikzcd} G \times_P (X' \times \mathfrak{n}) \arrow[rr, "\mu_G"] \arrow[dr, "\pi'"] & & \bar{\mathbb{O}} \subset \mathfrak{g} \\ & X' \arrow[ur] & \end{tikzcd}$$

Poisson b/c of this commut. diagr. b/c $X' \to \mathfrak{g}$ is unramified over points of $\tilde{\mathbb{O}} \subset X'$.

Also consider more gen-l Poisson vars: $\xi \in \mathfrak{z}(\mathfrak{l})$

$\rightsquigarrow \mu^{-1}(\xi)/L = G \times_P (\xi \times X' \times \mathfrak{n})$ – fiber in $\mu^{-1}(\mathfrak{z}(\mathfrak{l}))/L = G \times_P (\mathfrak{z}(\mathfrak{l}) \times X' \times \mathfrak{n})$

– Poisson family over $\mathfrak{z}(\mathfrak{l})$ & deform'n of $G \times_P (X' \times \mathfrak{n})$

Cor: X has sympl. sing-s

Proof: X' has $\Rightarrow G \times_P (X' \times \mathfrak{n})$ has $\Rightarrow$ [π' is Poisson] X has (just take resolution of $G \times_P (X' \times \mathfrak{n})$)

Rem: Ham. red'n construction shows that $(G \times_P (X' \times \mathfrak{n}))^{reg} = G \times_P (X'^{reg} \times \mathfrak{n})$ is symplc