# Lec 3: Deformations & orbit method

1) Filt'd deform'ns of sympl'c sing's

2) Primitive ideals & orbit method

**1.1) Filt'd deform'ns:** $A=\bigoplus_{i=0}^{\infty} A_i$ graded alg'a (commut've & assoc've) w. Poisson $\{\cdot,\cdot\}$ w $\deg\{\cdot,\cdot\}=-1$ ($\{A_i,A_j\}\subset A_{i+j-1}$), $A_0=\mathbb{C}$

Ex: $\tilde{\mathbb{O}}\subset G\times_P(\overline{\mathbb{O}}'\times\mathfrak{n})$ open $\rightsquigarrow A=\mathbb{C}[\tilde{\mathbb{O}}]$ (grading comes from $\mathbb{C}^\times$-action)

Def'n: Filt. Poisson deform'n of $A$ is $(\mathcal{A}^\circ,\iota)$

- $\mathcal{A}^\circ=\bigcup_{i\geq 0}\mathcal{A}^\circ_{\leq i}$ - filt. Poisson alg'a w. $\deg\{\cdot,\cdot\}\leq -1$ ($\{\mathcal{A}^\circ_{\leq i},\mathcal{A}^\circ_{\leq j}\}\subset\mathcal{A}^\circ_{\leq i+j-1}$)

$\rightsquigarrow \deg\{\cdot,\cdot\}=-1$ on $\operatorname{gr}\mathcal{A}^\circ=\bigoplus_i \mathcal{A}^\circ_{\leq i}/\mathcal{A}^\circ_{\leq i-1}$

- $\iota:\operatorname{gr}\mathcal{A}^\circ\xrightarrow{\sim}A$ (isom of graded Poisson alg's)

So we modify the product & Poisson bracket on $A$ by adding lower deg. terms so that the product is commut. & assoc & deformed $\{\cdot,\cdot\}$ is Poisson

Def'n: Filt. quantization of $(A,\iota)$

- $\mathcal{A}=\bigcup_{i\geq 0}\mathcal{A}_{\leq i}$ filt. assoc've (but not commut've) alg'a w. $\deg[\cdot,\cdot]\leq -1$

($\rightsquigarrow$ deg $-1$ $\{\cdot,\cdot\}$ on $\operatorname{gr}\mathcal{A}$ - commut.)

- $\iota:\operatorname{gr}\mathcal{A}\xrightarrow{\sim}A$ (of graded Poisson algebras)

We deform the product so that it's still associative, while the top degree of the commutator is the initial Poisson bracket.

Ex: Univ. envel. alg'a $\mathcal{U}(\mathfrak{g})=T(\mathfrak{g})/(x\otimes y-y\otimes x-[x,y]\,|\,x,y\in\mathfrak{g})$ ($\mathfrak{g}$ is Lie alg'a)

PBW Thm: $\mathcal{U}(\mathfrak{g})$ is filt. quant'n of $S(\mathfrak{g})=\mathbb{C}[\mathfrak{g}^*]$ - graded Poisson alg'a

Rem: Isom $(\mathcal{A}_1,\iota_1)\xrightarrow{\sim}(\mathcal{A}_2,\iota_2)$: filt. alg. isom $\varphi:\mathcal{A}_1\to\mathcal{A}_2$ w $\iota_1=\iota_2\circ(\operatorname{gr}\varphi)$

**1.2) Example - deform-ns of nilp. cone**

$\mathfrak{g}$ - s/simple $\supset\mathcal{N}$ subvar-y of nilp. el-ts

Kostant: $\mathcal{N}$ is irred-le, normal, given by ideal $\mathbb{C}[\mathfrak{g}]\,\mathbb{C}[\mathfrak{g}]^G_+$ inv. polyn-s w/o const. ter[m]

($\Rightarrow\mathcal{N}=\overline{\mathbb{O}}$ for princ. nilp orbit $\mathbb{O}$), $\mathbb{C}[\mathcal{N}]=\mathbb{C}[\mathbb{O}]$

Chevalley: $\mathbb{C}[\mathfrak{g}]^G\simeq\mathbb{C}[\mathfrak{h}]^W$, $\lambda\in\mathfrak{h}\rightsquigarrow\mathcal{A}^\circ_\lambda:=\mathbb{C}[\mathfrak{g}]/\mathbb{C}[\mathfrak{g}]\,\overline{\mathbb{C}[\mathfrak{g}]^G_\lambda}$ $:\{f\in\mathbb{C}[\mathfrak{g}]^G\,|\,f(\lambda)=$

- filt. Poisson (b/c $\mathbb{C}[\mathfrak{g}]^G$ is the Poisson center) deform'n of $\mathcal{A}^\circ_0=\mathbb{C}[\mathcal{N}]$

Harish-Chandra: center $Z$ of $U(\mathfrak{g}) \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}]^W \rightsquigarrow \mathcal{A}_\lambda = U(\mathfrak{g})/U(\mathfrak{g})Z_\lambda$

- filt. quant'n of $A = \mathbb{C}[\mathcal{N}]$

Concl-n: families of filt. Poisson deform'ns/quant'ns param. by $\mathfrak{h}/W$ ($\mathcal{A}_\lambda = \mathcal{A}_{w\lambda}$)

1.3 Classif-n of deform'ns of $\mathbb{C}[X]$, $X$ w. sympl sing's

$A = \mathbb{C}[X]$, $X$ has sympl. sing's

Thm 1 (Namikawa '09) $\exists$ fin. dim vect. space $\mathfrak{h}_X$ & $W_X \curvearrowright \mathfrak{h}_X$ crystall. refl-n gr-p (Namikawa-Weyl gr-p) s.t. $\{$filt. Poisson deform'ns of $A\}/\text{iso} \xrightarrow[\text{nat'l}]{\sim} \mathfrak{h}_X/W_X$

Thm 2 (I.L. '16) $\{$filt. quant-ns of $A\}/\text{iso} \xrightarrow[\text{nat'l}]{\sim} \mathfrak{h}_X/W_X$

reminder on 3.4

1.4 Spec. case $A = \mathbb{C}[\tilde{\mathbb{O}}]$: $\mathbb{O}' \subset \mathfrak{l}$ bir. rigid s.t. $\tilde{\mathbb{O}}$ is open in $G \times_P (X' \times \mathfrak{n})$

$\tilde{\mathbb{O}} = G/H$, $\overline{\mathbb{O}}$ := image of $\tilde{\mathbb{O}}$ in $\mathfrak{g}$, $e \in \mathbb{O} \rightsquigarrow Z_G(e)^\circ \subset H \subset Z_G(e)$

Thm 3 (I.L. 16) $\mathfrak{h}_X = \mathfrak{z}(\mathfrak{l})$ & $\exists$ s.e.s. $1 \to W_X \to N_G(\mathfrak{l}, \mathbb{O}')/L \to Z_G(e)/H \to 1$

Rem: The group $Z_G(e)/H$ is in fact the group of graded Poisson automorphisms of $A$. Also note that if $\mathbb{O}' = 0$, then we get a conceptual explanation of why $N_G(\mathfrak{l})/L$ acts on $\mathfrak{z}(\mathfrak{l})$ as a "small extension" of a crystall. refl-n group

Rem: $\xi \in \mathfrak{z}(\mathfrak{l}) \Rightarrow \mathcal{A}^0_\xi = \mathbb{C}[\tilde{\mathbb{O}}_\xi]$, $\tilde{\mathbb{O}}_\xi \subset G \times_P (\xi \times X' \times \mathfrak{n})$ open

Ex: $\mathbb{O}$ princ $\Rightarrow A = \mathbb{C}[\mathcal{N}]$, ind-d from $(\mathfrak{h}, 0) \Rightarrow \mathfrak{h}_X = \mathfrak{h}$, $W_X = W$, $\mathcal{A}^0_\lambda, \mathcal{A}_\lambda$ as before

2.1) Primitive ideals: $\mathfrak{g}$ - Lie alg, $\dim \mathfrak{g} < \infty \rightsquigarrow U(\mathfrak{g})$

Def: 2-sid. ideal $I \subset U(\mathfrak{g})$ is <u>primitive</u> if $\exists$ $U(\mathfrak{g})$-irrep $M$ s.t. $I = \mathrm{Ann}_{U(\mathfrak{g})}(M)$

One point to consider those is that there are way too many $\mathfrak{g}$-irreps and it's not possible to describe them all basically in any case. But the set of primitive ideals has a reasonable size. Here's a model result on classifying primitive ideals

$\mathrm{Prim}(\mathfrak{g}) = \{$prim-ve ideals in $U(\mathfrak{g})\}$

Thm (Dixmier '63) Let $\mathfrak{g}$ be nilp't. Then $\exists$ nat'l bij'n $\mathrm{Prim}(\mathfrak{g}) \xrightarrow{\sim} \mathfrak{g}^*/G$ (*)

Usually $\mathfrak{g}^*/G$ is quite nasty but, in principle, can be understood - think Donald Trump.

Now the question is what about s/simple $\mathfrak{g}$?

(*) This thm is an algebraic counterpart of Kirillov's Orbit method which ~~[illegible]~~ establishes a nat'l bijection between $\mathfrak{g}^*/G$ and unitary irreps of $G$

$\mathfrak{g}$-s/simple, $\lambda \in \mathfrak{h}^*_{77} \rightsquigarrow \mathfrak{g}$-irrep w. h.wt $\lambda \in \mathfrak{h}^*$, $L(\lambda)$

Thm (Duflo) $\forall\, \mathcal{J} \in \mathrm{Prim}(\mathfrak{g})\ \exists\ \lambda \in \mathfrak{h}^* \mid \mathcal{J} = \mathrm{Ann}_{U(\mathfrak{g})} L(\lambda)$

$\rightsquigarrow \mathfrak{h}^* \twoheadrightarrow \mathrm{Prim}(\mathfrak{g})$. Descr-n of fibers - via cells - from work of Joseph, Lusztig, Barbasch-Vogan in 80's. One thing to notice is that there's no direct connection with the set of orbits in G. Moreover, in a sense, the set of primitive ideals is bigger. However one can produce a nilpotent orbit from an ideal

$\mathcal{J} \subset U(\mathfrak{g}) \rightsquigarrow \mathrm{gr}\,\mathcal{J} \subset \mathbb{C}[\mathfrak{g}] \rightsquigarrow$ assoc. var-y $V(\mathcal{J}) \subset \mathfrak{g}$

Thm (Joseph $^{85}$) $\exists$ nilp. orbit $\mathbb{O}$ s.t. $V(\mathcal{J}) = \overline{\mathbb{O}}$.

2.2) Map $\mathfrak{g}/G \to \mathrm{Prim}(\mathfrak{g})$ (I.L. 11)

$\mathbb{O}_1 \in \mathfrak{g}/G$, $\xi = \xi_s + \xi_n$. Let $Z_G(\xi_s)\xi_n$ be birat. ind-d from $(\mathfrak{l}, \mathbb{O}')$, $\mathbb{O}'$ is bir. rigid

$\rightsquigarrow (\xi_s, \mathfrak{l}, \mathbb{O}') \rightsquigarrow \widetilde{\mathbb{O}}$ open in $G \times_P (X' \times \mathfrak{n})$, $A = \mathbb{C}[\widetilde{\mathbb{O}}]$, $\xi_s \in \mathfrak{z}(\mathfrak{l})$

$\mathbb{C}[\mathbb{O}_1] = \mathcal{A}^0_{\xi_s} \longleftrightarrow$ corresp. quant'n $\mathcal{A}_{\xi_s}$ $(=\mathcal{A})$

$\mathfrak{g} \to A_1$ (from $\widetilde{\mathbb{O}} \twoheadrightarrow \mathbb{O} \hookrightarrow \mathfrak{g}$) $\rightsquigarrow \mathfrak{g} \to \mathcal{A}_{\leq 1}$ $(\mathrm{gr}\,\mathcal{A} = A) \rightsquigarrow U(\mathfrak{g}) \to \mathcal{A}$

$\mathcal{J}_{\mathbb{O}_1} := \ker[U(\mathfrak{g}) \to \mathcal{A}]$

Rems: $V(\mathcal{J}_{\mathbb{O}_1}) = \overline{\mathbb{O}}$

The map is expected to be inj've $\forall \mathfrak{g}$. Proved for classical $\mathfrak{g}$.

See reverse for more