Lecture 4: $\mathbb{Q}$-factorial terminalizations & deformations.

1) $\mathbb{Q}$-factorial terminalizations

2) Deformations

We'll discuss how to construct filtered Poisson deform'ns & quant'ns of $\mathbb{C}[X]$ where X is affine w. symplectic singularities

1.1) $\mathbb{Q}$-factorial & terminal varieties: X normal alg. var'y

Def'n: X is $\mathbb{Q}$-factorial if $\forall$ Weil divisor D $\exists$ $m\in\mathbb{Z}\setminus\{0\} \mid mD$ is Cartier comes from a line bundle. E.g. smooth var'y is $\mathbb{Q}$-fact'l

Def'n: Let X be CM & Gorenstein ($\rightsquigarrow$ canon line bundle K_X) X has terminal sing's if $\forall$ resol'n $\rho:\tilde{X}\to X$ have $K_{\tilde{X}}=\rho^* K_X+\sum_i n_i E_i$ (E_i divisor, $n_i>0$)

The only fact about terminal sing's we'll be using is

Thm (Namikawa) Let X have sympl'c sing's. TFAE

- X has term'l sing's
- $\operatorname{codim}_X X^{sing}\geqslant 4$

1.2 $\mathbb{Q}$-factorial terminalizations

Appr. Thm (Birkar-Cassini-Hacon-McKernan) Under suitable assumptions on X ($\Leftarrow X$ has sympl'c sing's) $\exists(\hat{X},\rho:\hat{X}\to X)$ s.t

- $\hat{X}$ is $\mathbb{Q}$-factorial w. term sing's
- ρ is part'l resol'n (birat'l & proj've) and is crepant ($K_{\hat{X}}=\rho^*K_X$)

Cor: X ~~is sing~~ has sympl'c sing's $\Rightarrow$ so does $\hat{X}$

Proof: ω sympl. form on X^{reg}; $K_{\hat{X}}=\rho^*K_X\Rightarrow \rho^*\omega$ ext's to $\overbrace{\hat{\omega}}^{\text{sympl. form}}$ on $\hat{X}^{reg}$.

$\tilde{X}\xrightarrow[\tilde{\rho}]{}\hat{X}$ resol-n $\Rightarrow\tilde{\rho}^*\rho^*\omega$ ext's to $\tilde{X}\Rightarrow\tilde{\rho}^*\hat{\omega}$ ext's to $\tilde{X}$ $\square$

1.3. Cohomology vanishing: X affine w. sympl'c sing's, $\hat{X}$ $\mathbb{Q}$-factorial term'n

($\Rightarrow K_X\simeq\mathcal{O}_X, K_{\hat{X}}\simeq\mathcal{O}_{\hat{X}}$) - because X^{reg} has sympl'c form & $\operatorname{codim}_X X\setminus X^{reg}\geqslant 2$.

Grauert-Riemenschneider type thm: $R\rho_*\mathcal{O}_{\hat{X}}=\mathcal{O}_X$ ($\Leftrightarrow\mathbb{C}[\hat{X}]=\mathbb{C}[X]$ & $H^i(\hat{X},\mathcal{O}_{\hat{X}})=0$ $(i>0)$

4.1 Cor: $H^i(\hat{X}^{reg},\mathcal{O})=0$, $i=1,2$ & $\mathbb{C}[\hat{X}^{reg}]=\mathbb{C}[X]$ this just from normality

Proof: $\hat{X}$ is CM $\Leftrightarrow$ for $Y \subset \hat{X}$, $\operatorname{codim}_Y \hat{X} = d$ have $\mathcal{H}^i_Y(\mathcal{O}_{\hat{X}}) = 0$ for $i < d$.
From this & $H^i(\hat{X}, \mathcal{O}_{\hat{X}}) = 0\ \forall i \Rightarrow H^i(\hat{X} \setminus Y, \mathcal{O}_{\hat{X}}) = 0$ $i \leqslant d-2$. Now use $Y = \hat{X}^{sing}$ ($d \geqslant 4$).

1.4) Spec. case $X = \operatorname{Spec} \mathbb{C}[\tilde{O}]$: $\tilde{O} \subset G \times_P (X' \times \mathfrak{n})$, $X' = \operatorname{Spec} \mathbb{C}[O']$, O' is bir. rig
Thm (Namikawa - Fu, I.L) $\hat{X} = G \times_P (X' \times \mathfrak{n})$ is $\mathbb{Q}$-factorial terminalization of
Namikawa: for class. types, Fu for except. types - case by case; my proof is conceptual: it uses BCHM thm, classif'n of deformations & claim that $\mathbb{C}[O']$ has no filt'd Poisson deformations

2.1) Poisson deform'ns of $\hat{X}^{reg}$; $\mathfrak{h}_X := H^2(\hat{X}^{reg}, \mathbb{C})$
We want to study deformations of X. For this we need to study deformat of $\hat{X}^{reg}$ - and we will need formal deformations. The fact that $\hat{X}^{reg}$ is symplectic together with cohomology vanishing from Corollary show that the deformation theory is unobstructed i.e. as nice as possible
Y smooth sympl'c var'y w. form ω.
Def: B compl. local $\mathbb{C}$-algebra; formal deform'n of $(Y, \omega)/B$ is f.l scheme $\mathcal{Y}/\operatorname{Spec}(B)$ w. fiberwise sympl'c form $\tilde{\omega} \in \Omega^2(\mathcal{Y}/\operatorname{Spec}(B))$ s.t. spec'n of $(\mathcal{Y}, \tilde{\omega})$ to closed pt is (Y, ω).
Suppose $H^i(Y, \mathcal{O}_Y) = 0$, $i = 1, 2$
Thm (Kaledin-Verbitsky) $\exists$ univ. deform'n $(\mathcal{Y}_{univ}, \tilde{\omega}_{univ})$ over $B := \mathbb{C}[[H^2(Y, \mathbb{C})]]$
(every other deformation is obtained by pull-back)
Alternative way to think about $\mathcal{Y}$: $\mathcal{Y} \leftrightarrow$ complete (& separated) sheaf of Poisson B-alg's on Y ($\varphi: Y \hookrightarrow \mathcal{Y} \rightsquigarrow \varphi^{-1}\mathcal{O}_{\mathcal{Y}}$ -sheaf on Y) deforming $\mathcal{O}_Y$
sheaf theoretic pull-back

2.2) Deform'ns of X:
$H^i(Y, \mathcal{O}_Y) = 0$ $i = 1, 2 \Longrightarrow \mathbb{C}[\mathcal{Y}] = \Gamma(Y, \varphi^{-1}\mathcal{O}_{\mathcal{Y}})$ is filt. deform'n of $\mathbb{C}[Y] = \Gamma(Y, \mathcal{O}_Y)$
$Y = \hat{X}^{reg} \rightsquigarrow \mathcal{A}^{0\wedge}_{\mathfrak{h}} := \mathbb{C}[\mathcal{Y}] / \mathbb{C}[[\mathfrak{h}_X]]$ $\quad$ $\mathfrak{h}_X$ in deg 1
$\mathbb{C}^\times \curvearrowright \hat{X}^{reg}$ w. $t.\omega = t\omega \rightsquigarrow \mathbb{C}^\times \curvearrowright \mathcal{Y} \rightsquigarrow \mathbb{C}^\times \curvearrowright \mathcal{A}^{0\wedge}_{\mathfrak{h}} \rightsquigarrow \mathcal{A}^0_{\mathfrak{h}}$:= subalg. of $\mathbb{C}^\times$-finit
4.2 el-ts in $\mathcal{A}^{0\wedge}_{\mathfrak{h}}$; $A = \mathbb{C}[X]$ is posit. graded + $\mathfrak{h}$ in deg $1 \Longrightarrow \mathcal{A}^0_{\mathfrak{h}}$-graded deform'n

of $A/\mathbb{C}[\mathfrak{h}]$ ($\mathfrak{h}=\mathfrak{h}_X$)

~~Given~~ Now $W_X \curvearrowright \mathcal{A}^\circ_{\mathfrak{h}}$ (w. $W_X \curvearrowright A$ trivial & $W_X \curvearrowright \mathfrak{h}_X$ linear) preserving $\{\cdot,\cdot\}$

$\rightsquigarrow$ graded Poisson alg'a $(\mathcal{A}^\circ_{\mathfrak{h}})^{W_X}$

Thm (Namikawa) This is univ. graded Poisson deform'n of A

$\mathcal{A}^\circ$ is filt. Poisson deform'n, $\mathcal{A}^\circ = \bigcup_{i\geq 0} \mathcal{A}^\circ_{\leq i} \rightsquigarrow$ Rees alg'a $R_\hbar(\mathcal{A}^\circ) = \bigoplus \mathcal{A}^\circ_{\leq i}\hbar^i$

graded Poisson deform'n so obtained by pull-back from the universal one

$\Rightarrow \mathcal{A}^\circ = \mathcal{A}^\circ_\lambda$ for some $\lambda \in \mathfrak{h}$.

2.3) Quant'ns: Can talk about deformation quant'n of Poisson schemes (sheaves of $\mathbb{C}[[\hbar]]$-algebras)

Thm (Bezrukavnikov-Kaledin) ~~(...)~~ Under assumptions of Kaledin-Verbitsky Thm

$\exists$ canon. quant'n of $(Y_{univ}, \tilde{\omega}_{univ})$. Then we argue as before

2.4) Case of $A=\mathbb{C}[\tilde{\mathbb{O}}]$, $\hat{X} = G\times_P(\hat{X}\times\mathfrak{n}) = \mu_L^{-1}(0)/\!/L$, $\hat{X}_{\mathfrak{h}} = G\times_P(\mathfrak{z}(\mathfrak{l})\times\hat{X}\times\mathfrak{n}) =$

$=\mu^{-1}(\mathfrak{z}(\mathfrak{l}))/\!/L$ - Poisson scheme. So $\mathcal{A}^\circ_{\mathfrak{h}} = \mathbb{C}[\hat{X}_{\mathfrak{h}}]$, $\mathcal{A}^\circ_\lambda = \mathbb{C}[\tilde{\mathbb{O}}_\lambda]$, $\tilde{\mathbb{O}}_\lambda$ is open in $G\times_P(\lambda\times\hat{X}\times\mathfrak{n})$